Trends in Mathematics
Research Perspectives Ghent Analysis and PDE Center 6

Allaberen Ashyralyev
Michael Ruzhansky
Makhmud A. Sadybekov
Editors

Analysis and Applied Mathematics

Extended Abstracts of the 2022 Joint Seminar

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Research Perspectives Ghent Analysis and PDE Center

Volume 6

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Analysis and Applied Mathematics

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Preface

This book offers proceedings of the in-person and online "Analysis and Applied Mathematics" seminars organized jointly by the Bahcesehir University (Istanbul, Türkiye), Ghent Analysis and PDE Center (Ghent University, Ghent, Belgium), and the Institute of Mathematics and Mathematical Modeling (Almaty, Kazakhstan). This book of extended abstracts is part of our series Research Perspectives Ghent Analysis and PDE Center, devoted to the publication of abstracts, in an extended form, of talks presented at the events associated with the Ghent Analysis and PDE Center. We hope that this volume will be of value to professional mathematicians as well as advanced students in the fields of analysis and applied mathematics, providing an overview of some research topics in the wide area of analysis and their relevance to applied mathematics.

The goal of the joint seminar "Analysis and Applied Mathematics" is to provide a forum for researchers and scientists from different regions to communicate their recent developments and to present their original results in various fields of analysis and applied mathematics. The seminar originated in 2022, after the pandemic, mostly in the online format, to bring together mathematicians working in different institutions for discussions of joint topics of interest, fuelled by the work of the international community on these subjects. The website of the seminar can be found at https://sites.google.com/view/aam-seminars.

Many of the lectures given at the seminar have been recorded and are available on the YouTube Channel of the Institute of Mathematics of the University of Georgia. This includes many papers included in this volume, as well as other talks given at the seminar but do not appear here. The volume contains extended abstracts of these and a few related talks which were given at the seminar during the 2022–2023 period.

This book presents 23 papers by authors from different countries: Turkey, Kazakhstan, USA, Italy, Portugal, Spain, Serbia, Azerbaijan, Jordan, Lithuania, India, Iraq, Russian Federation, Uzbekistan, Tajikistan, and Turkmenistan. We are especially pleased with the fact that many articles are written by co-authors who work at different universities in the world. We are confident that such international integration provides an opportunity for a significant increase in the quality and quantity of scientific publications.

Publications in this book contain new results or overviews of some relevant mathematical areas. The volume reflects the latest developments in the area of analysis and applied mathematics and their interdisciplinary applications. This volume is organised in four parts. Part I contains the contributed papers focusing on various aspects of the analysis and its applications. Part II is devoted to the research on the theory of applied mathematics. Part III contains the results of studies on ordinary and partial differential equations and their applications. Finally, Part IV is focused on the simulation of problems arising in real-world applications of applied sciences.

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Istanbul, Turkey	Allaberen Ashyralyev
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Part I
Analysis

Chapter 1
Some Measures of Noncompactness and Their Applications

Eberhard Malkowsky

Abstract This is the extended abstract of the author's talk in the *Analysis and Applied Mathematics Weekly Online Seminar* on important results on measures of noncompactness, and some recent applications on the characterisations of compact operators between certain BK spaces, and in fixed point theorems.

1.1 Introduction

Measures of noncompactness are very useful tools in functional analysis, for instance, in metric fixed point theory, the characterisations of compact operators between Banach spaces, and the study of differential and integral equations.

We present an axiomatic introduction to measures of noncompactness on the class of bounded subsets of complete metric spaces, the definition and most important properties of the Kuratowski and Hausdorff measures of noncompactness, a study of measures of noncompactness of operators between Banach spaces, and some applications to the characterisations of compact linear operators between certain BK spaces and the solvability of an infinite system of integral equations.

Compactness and *measures of noncompactness* play an important role in fixed point theory. There are, however, cases when the operators are not compact and the results have to be extended to noncompact operators. Perhaps the most important application of a measure of noncompactness is *Darbo's fixed point theorem* [4], which uses *Kuratowski's measure of noncompactness* α [8]. Darbo's theorem is a generalisation of Schauder's fixed point theorem [17].

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1.2 Measures of Noncompactness

Measures of noncompactness are studied in detail and their use is discussed, for instance, in the monographs [1, 2, 9, 10, 18].

First, we recall the *axiomatic introduction* of the concept of a *measure of noncompactness* in complete metric spaces.

Definition 1.2.1 Let (X, d) be a complete metric space, and $\mathcal{M}_X$ be the class of bounded subsets of X. A set function $\phi : \mathcal{M}_X \to [0, \infty)$ that satisfies the following conditions for all $Q, Q_1, Q_2 \in \mathcal{M}_X$

(MNC.1) $\phi(Q) = 0$ if and only if Q is relatively compact (*regularity*)

(MNC.2) $\phi(Q) = \phi(\overline{Q})$ (*invariance under closure*)

(MNC.3) $\phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\}$ (*semi–additivity*)

is called a *measure of noncompactness on* $\mathcal{M}_X$ and $\phi(Q)$ is called the *measure of noncompactness of the set* Q.

Proposition 1.2.2 *Let (X, d) be a complete metric space. Any measure of noncompactness ϕ on $\mathcal{M}_X$ satisfies the following conditions for all $Q, Q_1, Q_2 \in \mathcal{M}_X$*

$$Q_1 \subset Q_2 \text{ implies } \phi(Q_1) \le \phi(Q_2) \qquad \text{(monotonicity)} \tag{1.1}$$

$$\phi(Q_1 \cap Q_2) \le \min\{\phi(Q_1), \phi(Q_2)\} \tag{1.2}$$

$$\phi(Q) = 0 \text{ for every finite set } Q \qquad \text{(non–singularity)}. \tag{1.3}$$

$$\left\{\begin{array}{l} \textit{If } (Q_n) \textit{ is a decreasing sequence of nonempty, closed sets in } \mathcal{M}_X \textit{ and} \\ \lim_{n\to\infty} \phi(Q_n) = 0, \ \textit{then} \\ \qquad\qquad Q_\infty = \bigcap_{n=1}^{\infty} Q_n \neq \emptyset \textit{ is compact} \\ \text{(Cantor's generalised intersection property [18, p. 19]);} \\ \text{([8, 1930] for } \phi = \alpha.) \end{array}\right\} \tag{1.4}$$

Now we recall the definitions of the *Kuratowski* and *Hausdorff measures of noncompactness* in complete metric spaces (X, d).

Definition 1.2.3

(a) ([8] or [18, Definition II.2.1]) The *Kuratowski measure of noncompactness* is the map $\alpha : \mathcal{M}_X \to [0, \infty)$ with

$$\alpha(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} S_k, \ S_k \subset X, \right.$$
$$\left. \operatorname{diam}(S_k) < \varepsilon \ (k = 1, 2, \dots, n \in \mathbb{N})\right\}.$$

(b) ([21] or [18, Definition II.2.1]) The *Hausdorff* or *ball measure of noncompactness* is the map $\chi : \mathcal{M}_X \to [0, \infty)$ with

$$\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{k=1}^{n} B_{r_k}(x_k),\; x_k \in X, \right.$$
$$\left. r_k < \varepsilon \; (k = 1, 2, \dots, n \in \mathbb{N}) \right\},$$

where $B_{r_k}(x_k)$, as usual, denotes the open ball of radius r_k and centre in x_k.

Remark 1.2.4 We note that the functions α and χ are measures of noncompactness in the sense of Definition 1.2.1. So they satisfy (1.1)–(1.4) ([9, Lemmas 2.6, 2.11, Theorem 2.7] and [18, Remark 3.2]). They are also equivalent ([18, Remark 3.2]), that is, $\chi(Q) \leq \alpha(Q) \leq 2 \cdot \chi(Q)$ for all $Q \in \mathcal{M}_X$. Studies on inequivalent measures of noncompactness can be found, for instance, in [12, 13].

Some measures of noncompactness such as α and χ satisfy several important conditions that are connected to the linear structure of Banach spaces; the statements for α in (1.5)–(1.8) of Proposition 1.2.5 are due to Darbo [4].

Proposition 1.2.5 ([10, Theorems 7.6.7, 7.7.6 (b)]) *Let X be a Banach space, $Q, Q_1, Q_2 \in \mathcal{M}_X$, ψ be any of the functions α or χ, and* co(Q) *denote the convex hull of Q. Then we have*

$$\psi(Q_1 + Q_2) \leq \psi(Q_1) + \psi(Q_2) \qquad \text{(sublinearity)}, \tag{1.5}$$

$$\psi(Q + x) = \psi(Q) \textit{ for each } x \in X \qquad \text{(translation invariance)}, \tag{1.6}$$

$$\psi(\lambda Q) = |\lambda| \psi(Q) \textit{ for each scalar } \lambda \qquad \text{(absolute homogeneity)} \tag{1.7}$$

$$\psi(Q) = \psi(co(Q)) \qquad \text{(invariance under passage to the convex hull)}. \tag{1.8}$$

If X is infinite dimensional, and B_X and S_X denote the open unit ball and the unit sphere in X, then $\alpha(B_X) = \alpha(S_X) = 2$ and $\chi(B_X) = \chi(S_X) = 1$ ([9, Theorems 2.9, 2.14]).

As an application of the results concerning measures of noncompactness we are going to state the famous theorem by *Goldenštein, Go'hberg and Markus*, which establishes an estimate for the Hausdorff measure of compactness of bounded sets in any Banach space with a Schauder basis.

Theorem 1.2.6 (Goldenštein, Go'hberg, Markus) (R-BIB.GGM1 *or* [18, Theorem II.4.2] *or* [9, Theorem 2.23])
Let X be a Banach space with a Schauder basis (b_k). Then the function $\mu : \mathcal{M}_X \to [0, \infty)$ defined by

$$\mu(Q) = \limsup_{n\to\infty} \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \textit{ with } \mathcal{R}_n(x) = \sum_{k=n+1}^{\infty} \lambda_k b_k \tag{1.9}$$

for all $x = \sum_{k=0}^{\infty} \lambda_k b_k \in X$ *satisfies the following inequality for every* $Q \in \mathcal{M}_X$

$$\frac{1}{a} \cdot \mu(Q) \leq \chi(Q) \leq \mu(Q),$$

where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$ *is the basis constant.*

So far, we measured the noncompactness of bounded subsets of complete metric spaces and Banach spaces. Now we introduce the concept of measures of noncompactness of operators between Banach spaces.

Definition 1.2.7 ([9, Definition 2.24]) Let ϕ_1 and ϕ_2 be measures of noncompactness on the Banach spaces X and Y, respectively. An operator $T : X \to Y$ is said to be (ϕ_1, ϕ_2)*–bounded* if $T(Q) \in \mathcal{M}_Y$ for each $Q \in \mathcal{M}_X$, and there exists a real number $k > 0$ such that $\phi_2(T(Q)) \leq k\phi_1(Q)$ for each $Q \in \mathcal{M}_X$.
If an operator T is (ϕ_1, ϕ_2)–bounded, then $\|T\|_{\phi_1,\phi_2}$ defined by

$$\|T\|_{\phi_1,\phi_2} = \inf\{k \geq 0 : \phi_2(T(Q)) \leq k\phi_1(Q) \text{ for each } Q \in \mathcal{M}_X\}$$

is called (ϕ_1, ϕ_2)*–operator norm of* T, or (ϕ_1, ϕ_2)*–measure of noncompactness of* T, or simply *measure of noncompactness of* T.
If $\phi_1 = \phi_2 = \phi$, then we write $\|T\|_\phi$ instead of $\|T\|_{\phi,\phi}$.

Theorem 1.2.8 *Let* X *and* Y *be Banach spaces,* $L \in \mathcal{B}(X, Y)$*,* S_X *and* $\overline{B}_X$ *be the unit sphere and the closed unit ball in* X.

(a) ([9, Theorem 2.25]) *Then we have* $\|L\|_\chi = \chi(L(S_X)) = \chi(L(\overline{B}_X))$.
(b) ([9, Corollary 2.26]) *Let* $\mathcal{C}(X, Y)$ *be the set of all compact operators in* $\mathcal{B}(X, Y)$. *Then* $\|\cdot\|_\chi$ *is a seminorm on* $\mathcal{B}(X, Y)$*,*

$$\|L\|_\chi = 0 \text{ if and only if } L \in \mathcal{C}(X, Y), \tag{1.10}$$

and $\|L\|_\chi \leq \|L\|$.

Important applications of the theory of measures of noncompactness are Darbo's fixed point theorem and its generalisation, the *Darbo–Sadovskiĭ theorem*. The important hypotheses are the condensing property (1.11), the invariance of the passage to the convex hull (1.8) of the measures of noncompactness involved, and Cantor's generalised intersection property (1.4).

Theorem 1.2.9 (Darbo's Fixed Point Theorem) ([4]) *Let* C *be a non–empty bounded, closed and convex subset of a Banach space* X *and* α *be the Kuratowski measure of noncompactness on* X. *If* $f : C \to C$ *is continuous such that there exists a constant* $c \in [0, 1)$ *with*

$$\alpha(f(Q)) \leq c \cdot \alpha(Q) \text{ for every } Q \subset C, \tag{1.11}$$

then f *has a fixed point in* C.

Theorem 1.2.10 (Darbo–Sadovskiĭ) (R-BIB.Sad2, [18, Theorem 5.4, p. 40] or [10, Theorem 7.10.3])
Let X be a Banach space, ϕ be a measure of noncompactness which is invariant under passage to the convex hull, $C \neq \emptyset$ be a bounded, closed and convex subset of X and $f : C \to C$ be an operator that satisfies the condensing property (1.11), with ϕ in place of α. Then f has a fixed point in C.

1.3 Some Applications

Here we apply the results of Sect. 1.2 to the characterisations of some classes of bounded linear and compact operators on the *generalised Hahn space* h_d, and give a generalisation of Darbo's fixed point theorem and its application to the solution of an integral equation. We recommend [2] and [3] for further comprehensive studies of applications of measures of noncompactness to the solvability of infinite systems of differential and integral equations.

We use the standard notations ω, ℓ_∞ and c_0 for the sets of all complex, bounded and null sequences $x = (x_k)_{k=1}^\infty$; bs and $bv = \{x \in \omega : \sum_{k=1}^\infty |x_k - x_{k+1}| < \infty\}$, for the sets of all bounded series, and of all series of bounded variation. We also write $bv_0 = bv \cap c_0$. If $m \in \mathbb{N}$ and $x = (x_k)_{k=1}^\infty \in \omega$, then we write $x^{[m]} = (x_k^{[m]})_{k=1}^\infty$ for the *m–section of* x, where $x_k^{[m]} = x_k$ for $1 \le k \le m$ and $x_k^{[m]} = 0$ for $k > m$.

We refer the reader to [10, Definitions 9.2.1 and 9.2.12] for the concepts and fundamental properties of BK and AK spaces.

Let $d = (d_k)_{k=1}^\infty$ be a given monotone increasing unbounded sequence of positive real numbers. For every sequence $x = (x_k)_{k=1}^\infty \in \omega$, let $\Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty$ be the sequence of the forward differences of the sequence x. The *generalised Hahn space* is defined as [6]

$$h_d = \left\{ x = (x_k)_{k=1}^\infty \in \omega : \sum_{k=1}^\infty d_k |\Delta x_k| < \infty \right\} \cap c_0.$$

If $d_k = k$ for all k, then $h_d = h$, the original Hahn space h [7, 1922], and if $d = e = (1, 1, \dots)$, then $h_e = bv_0$.

Since h_d is a BK space with AK by Malkowsky et al. [11, Proposition 2.1], every $L \in \mathcal{B}(h_d) = \mathcal{B}(h_d, h_d)$ is given by an infinte matrix $A = (a_{nk})_{n,k=1}^\infty$ such that $L(x) = Ax = (A_n(x))_{n=1}^\infty$ for all sequences $x = (x_k)_{k=1}^\infty$, where $A_n x = \sum_{k=1}^\infty a_{nk} x_k$ for all $n \in \mathbb{N}$, and conversely, if $Ax \in h_d$ for all $x \in h_d$, then $L_A \in \mathcal{B}(h_d)$, where $L_A x = Ax$ for all $x \in h_d$ ([10, Theorem 9.3.3]).

First, we need to characterise the class $\mathcal{B}(h_d)$ and deternmine the operator norm of $L \in \mathcal{B}(h_d)$.

Theorem 1.3.1 ([11, Theorem 3.9 and Corollary 3.15 (a)]) *We have $L \in \mathcal{B}(h_d)$ if and only if $Ax = L(x) \in h_d$ for all $x \in h_d$ and this is the case if and only if*

$$\lim_{n\to\infty} a_{nk} = 0, \text{ for all } k, \tag{1.12}$$

and

$$\|A\|_{(h_d,h_d)} = \sup_m \left(\frac{1}{d_m} \sum_{n=1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| \right) < \infty. \tag{1.13}$$

If $L \in \mathcal{B}(h_d)$, then

$$\|L\| = \|A\|_{(h_d,h_d)}. \tag{1.14}$$

Proof (Outline) The proof uses the concept of *determining sets for BK spaces* ([19, Definition 7.4.2]), [11, Propositions 3.2 and 2.3] and [19, Theorem 8.3.4].

(i) First we note that, by Malkowsky et al. [11, Proposition 3.2], $E = \left\{(1/d_k) \cdot e^{[k]} : k \in \mathbb{N}\right\}$ is a determining set for h_d. Also, by Malkowsky et al. [11, Proposition 2.3], the continuous dual h_d^* of h_d is normisomorphic to $bs_d = \{a \in \omega : \sup_n (1/d_n)|\sum_{k=1}^n a_k| < \infty\}$ with the natural norm $\|a\|_{bs_d} = \sup_n (1/d_n)|\sum_{k=1}^n a_k|$ for all $a \in bs_d$.

(ii) Writing $y^{[m]} = (1/d_m) \cdot e^{[m]}$ for all $m \in \mathbb{N}$, we show $\sup_m \|Ay^{[m]}\|_{bs} < \infty$ and $Ay^{[m]} \in c_0$ for all $y^{[m]} \in E$. We note that the first condition is (1.12) and the seond condition is equivalent to (1.13). Hence we have obtained Condition (ii) in [19, Theorem 8.3.4]. Also condition (i) in [19, Theorem 8.3.4] is redundant, since the columns $A^k = (a_{nk})_{n=1}^{\infty}$ of A are in c_0 for each k by (1.12), and

$$\|A^k\|_{h_d} \leq d_k \|Ay^{[k]}\|_{h_d} + d_{k-1} \|Ay^{[k-1]}\|_{h_d} < \infty$$

for all k. Thus we obtain the characterization of $\mathcal{B}(h_d)$.

(iii) We obtain $\|L(x)\|_{h_d} \leq \|A\|_{(h_d,h_d)} \|x\|_{h_d}$ for all $x \in h_d$, so $\|L\| \leq \|A\|_{(h_d,h_d)}$. Conversely $\|L(y^{[m]})\|_{h_d} \leq \|L\|$ for all m yields $\|A\|_{(h_d,h_d)} \leq \|L\|$. This yields (1.14).

□

An application of Theorem 1.3.1 yields the multiplier $M(h_d, h_d)$, and the value a of the basis constant for h_d. We recall that the multiplier of $X \subset \omega$ in $Y \subset \omega$ is the set

$$M(X, Y) = \left\{z \in \omega : z \cdot x = (z_k x_k)_{k=1}^{\infty} \in Y \text{ for all } x = (x_k)_{k=1}^{\infty} \in X\right\}.$$

We also obtain the value of the basis constant a of h_d.

Example 1.3.2 ([11, Remark 4.6])

(a) It follows from Theorem 1.3.3 that

$$M(h_d, h_d) = \left\{ z \in \omega : \left(\frac{1}{d_m} \cdot \left\| z^{[m-1]} \right\|_{h_d} \right)_{m=1}^{\infty} \in \ell_\infty \right\}.$$

(b) Let $l \in \mathbb{N}$ be given, $(c_m^{(l)})_{m=1}^{\infty}$ be the sequence with $c_m^{(l)} = 0$ for $1 \leq m \leq l$ and $c_m^{(l)} = 1 + d_l/d_m$ for $m \geq l+1$, then

$$a = \limsup_{l \to \infty} \|\mathcal{R}_l\| = \limsup_{l \to \infty} \left(\sup_{m \geq l} c_m^{(l)} \right) = \limsup_{l \to \infty} \left(\sup_{m \geq l} \left(1 + \frac{d_l}{d_m} \right) \right) = 2.$$

Now we use Theorem 1.3.1 to establish an estimate for $\|L\|_\chi$ for every $L \in \mathcal{B}(h_d)$.

Theorem 1.3.3

(a) ([11, Theorem 4.8 (a)]) *Let $L \in \mathcal{B}(h_d)$. We write*

$$\gamma_m^{<l>} = \frac{1}{d_m} \left(d_l \left| \sum_{k=1}^{m} a_{l+1,k} \right| + \sum_{n=l+1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| \right)$$

for all m and l. Then we have

$$\frac{1}{2} \cdot \limsup_{l \to \infty} \left(\sup_m \gamma_m^{<l>} \right) \leq \|L\|_\chi \leq \limsup_{l \to \infty} \left(\sup_m \gamma_m^{<l>} \right). \tag{1.15}$$

(b) ([11, Theorem 4.10 (d)]) *We have $L \in C(h_d) = C(h_d, h_d)$ if and only if*

$$\lim_{l \to \infty} \left(\sup_m \gamma_m^{<l>} \right) = 0.$$

Proof (Outline) Let A be an infinite matrix with the rows A_n ($n \in \mathbb{N}$). For each $m \in \mathbb{N}$, we write $A^{<m>}$ for the matrix with the rows $A_n^{<m>} = 0$ for $n \leq m$ and $A_n^{<m>} = A_n$ for $n \geq m+1$. Also let $L^{<m>}$ denote the operator represented by $A^{<m>}$. Obviously $L^{<m>} = \mathcal{R}_m \circ L$ ($m \in \mathbb{N}$) for $L \in \mathcal{B}(h_d)$. First we have by Theorem 1.3.1 for all l

$$\|L^{<l>}\| = \sup_m \left(\frac{1}{d_m} \sum_{n=1}^{\infty} d_n \left| \sum_{k=1}^{m} (a_{nk}^{<l>} - a_{n+1,k}^{<l+1>}) \right| \right) = \sup_m \gamma_m^{<l>}.$$

Since $a = 2$ by Example 1.3.2 (b), (1.9) yields the inequalities in (1.15). Finally, Part (b) follows from (1.15) and (1.10). □

We apply Theorem 1.3.3 and Example 1.3.2 (a) to obtain two results by *Sawano* and *El–Shabrawy* [16, Corollary 5.1 and Lemma 5.1].

Rhaly [14] introduced the generalised Cesàro operator C_t on ω for $t \in [0, 1)$ by the matrix $C_t = (a_{nk}(t))_{n,k=0}^{\infty}$ with $a_{nk} = t^{n-k}/(n+1)$ for $(0 \leq k \leq n)$ and $a_{nk} = 0$ for $k > n$ $(n = 0, 1, \ldots)$.

Example 1.3.4 ([16, Corollary 5.1]) Let $0 \leq t < 1$. Then $L_{C_t} \in \mathcal{B}(h)$.

The special case of $d_k = k$ for all k of the next example yields [16, Lemma 5.1].

Example 1.3.5 ([5, Example 10]) Let $(\lambda_k)_{k=1}^{\infty}$ be a decreasing sequence of positive real numbers which converges to 0 and $D(\lambda) = diag(\lambda_1, \lambda_2, \ldots)$ denote the diagonal matrix with the sequence λ on its diagonal. Then $L_{D(\lambda)} \in C(h_d)$.

We also give an application of our results to Fredholm operators. We recall the definition of Fredholm operators ([10, Definition 8.4.1]). Let X and Y be Banach spaces, $L \in \mathcal{B}(X, Y)$, and $N(L)$ and $R(L)$ denote the null space and the range of L, respectively. Then L is said to be a *Fredholm operator*, if $R(L)$ is closed, and both dimensions $\dim N(T)$ and $\dim X/R(L)$ are finite. The *index* of a Fredholm operator L is defined as $i(L) = \dim N(L) - \dim X/R(L)$. Let us recall that if $L \in \mathcal{B}(X)$ and $\|L\|_{\chi} < 1$, then $I - L$ is a Fredholm operator and $i(I - L) = 0$ ([20] or [10, Section 7.13]).

Corollary 1.3.6 ([11, Corollary 4.11]) *Let $\alpha = (\alpha_n)_{n=1}^{\infty}$, $\beta = (\beta_n)_{n=1}^{\infty}$ and $\gamma = (\gamma_n)_{n=1}^{\infty}$ be given sequences of complex numbers, and $A(\alpha, \beta, \gamma)$ denote the tridiagonal matrix with α on the main diagonal, γ on the subdiagonal and β on the diagonal above the main diagonal.*
Then the operator $L \in \mathcal{B}(h_d)$ represented by the matrix $A(\gamma, \alpha, \beta) = A(0, \alpha, 0) + A(\gamma, 0, 0) + A(0, 0, \beta)$ is Fredholm with index $i(A(\alpha, \beta, \gamma)) = 0$ if $A(0, \alpha, 0)$ is Fredholm with index $i(A(0, \alpha, 0)) = 0$ and $A(\gamma, 0, 0)$ and $A(0, 0, \beta)$ are compact.

Example 1.3.7 ([11, Example 4.12]) Let $d_k = k$, $\alpha_k = 1 - 1/k$ and $\beta_k = \gamma_k = 1/k$ for all k. Then the operator $L \in \mathcal{B}(h_d)$ represented by the matrix $A(\gamma, \alpha, \beta)$ is Fredholm.

Finally, we consider a generalisation of Darbos's fixed point theorem, Theorem 1.2.9, and its application to the existence of solutions of a functional integral equation of Volterra type [15, Theorem 3.1]. We need the following definition.

Definition 1.3.8 ([15, Definition 2.1]) Let X be a Banach space and ϕ be a measure of noncompactness on $\mathcal{M}_X$ which is invariant under the passage to the convex hull (1.8), and homogeneous (1.7). Furthermore, let $H : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing map such that, for each sequence (a_n) of positive real numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} H(a_n) = 0$. A map $T : X \to X$ is said to be a *countable H–set contraction* if there exists a $\tau > 0$ such that, for all countable $Q \in \mathcal{M}_X$, $\phi(T(Q)) > 0$ implies $\tau + H(\phi(T(Q))) \leq H(\phi(Q))$.

The next result generalises Darbo's fixed point theorem.

Theorem 1.3.9 ([15, Theorem 2.8]) *Let C be a non–empty, bounded, closed and convex subset of a Banach space X, ϕ be a measure of noncompactness (as above) and $T : X \to X$ be a continuous H–contraction. Then T has a fixed point.*

An application of Theorem 1.3.9 yields a result on the solvability of the nonlinear integral equation

$$x(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))\, ds \ \ (t \in \mathbb{R}^+) \tag{1.16}$$

in the space $\mathcal{BC}(\mathbb{R}^+)$ which consists of all real functions defined continuous and bounded on $\mathbb{R}^+$; the norm on $\mathcal{BC}(\mathbb{R}^+)$ is defined by $\|x\| = \sup_{t\in\mathbb{R}^+}\{|x(t)|\}$.

Theorem 1.3.10 ([15, Theorem 3.1]) *We consider the following conditions:*

(i) *The function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous, but, for any nonempty bounded subset X of $\mathcal{BC}(\mathbb{R}^+)$, the family $\{f(t, x) : x \in X\}$ is equi–continuous for all $t \in \mathbb{R}^+$, and the function $t \mapsto f(t, 0)$ is a member of the space $\mathcal{BC}(\mathbb{R}^+)$. Moreover, there exists $\tau > 0$ such that*

$$|f(t, x)-f(t, y)| \neq 0 \text{ implies } \tau+H(|f(t, x)-f(t, y)|) \leq H(|x-y|).$$

(ii) *The function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist continuous functions $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $|g(t, s, x)| \leq a(t)b(s)$ for all $t, s \in \mathbb{R}^+$ with $s \leq t$ and $x \in \mathbb{R}$, where $\lim_{t\to\infty} a(t) \int_0^t b(s)\, ds = 0$.*

(iii) *There exists a positive solution r_0 of the inequality $H^{-1}(H(r_0) - \tau) + q \leq r_0$, where q is the constant defined by $q = \sup_{t\geq 0} \left\{|f(t, 0)| + a(t) \int_0^t b(s)\, ds\right\}$.*

Let (i), (ii) *and* (iii) *be satisfied. Then the nonlinear integral equation (1.16) has at least one solution in the space $\mathcal{BC}(\mathbb{R}^+)$.*

References

1. Akhmerov, R.R., Kamenskii, M.I., Potapov, A.S., Rodkina, A.E., Sadovskii, B.N.: Measures of noncompactness and condensing operators. In: Operator Theory. Advances and Applications, vol. 55. Springer, Basel (1992)
2. Banaś, J., Goebel, K.: Measures of Noncompactness in Banach Spaces. Lecture Notes in Pure and Applied Mathematics, vol. 60. Marcel Dekker, New York and Basel (1980)
3. Banaś, J., Jleli, M., Mursaleen, M., et. al. (eds.): On some results using measures of noncompactness. In: Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness. Springer, Berlin (2017)
4. Darbo, G.: Punti uniti in transformazioni a condomio non compatto. Rend. Sem. Math. Univ. Padova **24**, 84–92 (1955)
5. Gabeleh, M., et al.: A survey of measures of noncompactness and their applications. Axioms **13**, 367 (2024). https://www.mdpi.com/journal/axioms

6. Goes, G.: Sequences of bounded variation and sequences of Fourier coefficients II. J. Math. Anal. Appl. **39**, 477–494 (1972)
7. Hahn, H.: Über Folgen linearer operationen. Monatsh. Math. Phys. **32**, 3–88 (1922)
8. Kuratowski, K.: Sur les espaces complets. Fund. Math. **15**, 301–309 (1930)
9. Malkowsky, E., Rakočević, V.: An introduction into the theory of sequence spaces and measures of noncompactness. Zbornik Radova **9**(17), 143–234 (2000). Four Topics in Mathematics. Matematički institut SANU, Belgrade
10. Malkowsky, E., Rakočević, V.: Advanced Functional Analysis. Taylor and Francis, Boca Raton (2019)
11. Malkowsky, E., Rakočević, V., Tuğ, O.: Compact operators on the Hahn space. Monatsh. Math. **196**(3), 519–551 (2021)
12. Mallet-Paret, J., Nussbaum, R.D.: Inequivalent measures of nonccompactness. Ann. Mat. **190**, 453–488 (2011)
13. Mallet-Paret, J., Nussbaum, R.D.: Inequivalent measures of nonccompactness and the radius of the essential spectrum. Proc. Am Math. Soc. **193**(3), 917–930 (2011)
14. Rhaly, H.C.: Discrete generalized Cesàro operators. Proc. Am. Math. Soc. **86**(3), 405–409 (1982)
15. Salahifard, R., Vaezpour, S.M., Malkowsky, E.: Generalized Darbo's theorem and its applications. J. Nonlinear Convex Anal. **16**(8–10), 1–9 (2015)
16. Sawano, Y., El-Shabrawy, S.R.: Fine spectra of the discrete generalized Cesàro operator on Banach sequence spaces. Monatsh. Math. **192**, 185–224 (2020)
17. Schauder, J.: Der Fixpunktsatz in Funktionalräumen. Stud. Math. **2**, 171–180 (1930)
18. Toledano, J.A., Benavides, T.D., Acedo, G.L.: Measures of Noncompactness in Metric Fixed Point Theory. Operator Series, Advances and Applications, vol. 99. Birkhäuser, Basel (1997)
19. Wilansky, A.: Summability Through Functional Analysis. North-Holland Mathematics Studies, vol. 85. North-Holland, Amsterdam (1984)
20. Л. С. Гольденштейн, И. Ц. Гохберг и А. С. Маркус, Исследование некоторых свойств линейных ограниченных операторов в связи с их q-нормой, *Уч. зап. Кишиневского гос. ун-та* **29** (1957) 29–36
21. Л. С. Гольденштейн, А. С. Маркус, О мере некомпактности ограниченных множеств и линейных операторов, *В кн.: Исследование по алгебре и математическому анализу, Кишинев: Картя Молдавеняске* (1965) 45–54
22. Б. Н. Садовскй, Предельно компактные и уплотняющие операторы, *Успехи мат. наук*, **27** (1972) 81–146.

Chapter 2
Definition of Hessians for $m-$Convex Functions as Borel Measures

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Abstract In this work, $m-$convex functions are defined in the class of bounded upper semi-continuous functions of real arguments and a connection is established between $m-$convex and well-known violent $m-$subharmonic functions. As a consequence, we define in the class of $m-$convex functions, the Hessians H^k, $k = 1, 2, \dots, n - m + 1$, as Borel measures.

2.1 Introduction

$m-$convex functions in $\mathbb{R}^n$ are a real analogue of violent $m-$subharmonic (sh_m) functions in complex space $\mathbb{C}^n$. Let us recall the definition of a class of sh_m-functions, which has become the subject of research by many authors (Blocki [1], Dinev and Kolodziej [2], Li [3], Lu [4, 5], Abdullaev and Sadullaev [6, 7], etc.).

A twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is called violent subharmonic $u \in sh_m(D)$, if at each point of the domain D

$$\begin{gathered} sh_m(D) = \left\{u \in C^2 : \ (dd^c u)^k \wedge \beta^{n-k} \geq 0, \ \ k = 1, 2, \dots, n - m + 1\right\} \\ = \Big\{u \in C^2 : \ dd^c u \wedge \beta^{n-1} \geq 0, (dd^c u)^2 \wedge \beta^{n-2} \geq 0, \dots, \ (dd^c u)^{n-m+1} \wedge \\ \beta^{m-1} \geq 0\Big\}, \end{gathered} \tag{2.1}$$

where $\beta = dd^c \left\|z\right\|^2$ —is the standard volume form in $\mathbb{C}^n$.

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Operators $(dd^c u)^k \wedge \beta^{n-k}$ are closely related to Hessians. For a doubly smooth function $u \in C^2(D)$, the second order differential

$$dd^c u = \frac{i}{2} \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

(at a fixed point $o \in D$) is Hermitian quadratic form. After a suitable unitary coordinate transformation, it is reduced to diagonal form

$$dd^c u = \frac{i}{2} [\lambda_1 dz_1 \wedge d\bar{z}_1 + \ldots + \lambda_n dz_n \wedge d\bar{z}_n],$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hermitian matrix $(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k})$, which are real: $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Note that the unitary transformation does not change the differential form $\beta = dd^c \|z\|^2$. Therefore, it is easy to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)! H_o^k(u) \beta^n,$$

where $H_o^k(u) = \sum\limits_{1 \le j_1 < \ldots < j_k \le n} \lambda_{j_1} \ldots \lambda_{j_k}$ is the Hessian of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$ of dimension k.

Consequently, a doubly smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is violent $m-$subharmonic if at each point $o \in D$ we have

$$H^k(u) = H_o^k(u) \ge 0, \quad k = 1, 2, \ldots, n-m+1. \tag{2.2}$$

Note that the concept of a violent m-subharmonic function in the generalized sense is determined in the general case.

Definition 2.1 A function $u \in L^1_{loc}(D)$ is called sh_m in the domain $D \subset \mathbb{C}^n$, if it is upper semi-continuous, $u(z) \ge \overline{\lim\limits_{w \to z}} u(w)$ $\forall z \in D$ and for any doubly smooth sh_m functions $v_1, \ldots, v_{n-m} \in C^2(D) \bigcap sh_m(D)$ the following

$$dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1},$$

defined as

$$\begin{aligned} &\left[dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1}\right](\omega) \\ &= \int u \, dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \qquad \omega \in F^{0,0} \end{aligned} \tag{2.3}$$

is positive.

Blocki in the work [1] proved that this definition is correct, that for $u \in C^2(D)$ functions this definition coincides with the original definition of sh_m-functions. Moreover, in the class of bounded sh_m-functions, the operators

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, \quad k = 1, 2, \dots, n - m + 1$$

are defined as Borel measures in the domain D (see [1, 6]).

2.2 $m-$ Convex Functions

Now let $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D)$. Similar to (2.2), we want to define $m-$convex functions in the domain $D \subset \mathbb{R}^n$. The matrix $(\frac{\partial^2 u}{\partial x_j \partial x_k})$ is orthogonal, i.e., $\frac{\partial^2 u}{\partial x_j \partial x_k} = \frac{\partial^2 u}{\partial x_k \partial x_j}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right) \to \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $(\frac{\partial^2 u}{\partial x_j \partial x_k})$. Let $H_k(u) = H_k(\lambda) = \sum\limits_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j\,1} \dots \lambda_{j_k}$ be the Hessians of $k-$ dimensional of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.

Definition 2.2 A twice smooth function $u \in C^2(D)$ is called $m-$convex in $D \subset \mathbb{R}^n$, $u \in m-cv(D)$, if its eigenvalue vectors $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$ satisfy the conditions

$$m - cv \bigcap C^2(D) = \{H_k(u) = H_k(\lambda(x)) \geq 0, \ \forall x \in D, \ \ k = 1, \dots, n - m + 1\}$$

at each point $o \in D$.

Function theory of $m-cv$ is not studied much and is a new direction in the theory of real geometry. However, when $m = 1$ this class

$$1 - cv \bigcap C^2(D) = \{H_1(\lambda) \geq 0\} = \{\lambda_1 \geq 0, \dots, \lambda_n \geq 0\}$$

coincides with the class of convex functions in $\mathbb{R}^n$, and when $m = n$ the class $n - cv \bigcap C^2(D) = \{\lambda_1 +, \dots, \lambda_n \geq 0\}$ coincides with the class of subharmonic, (sh) functions. The class of convex functions has been well studied (Aleksandrov [8, 9], Bakelman [10, 11], Pogorelov [12], Artykbaev [20], etc.). For $m > 1$ this class was studied in a series of works by N. Ivochkina, N. Trudinger, X. Wang, S. Li, H. Lu et al. (see [3–5, 13–19].)

The principle difficulty in the theory of $m-cv$ functions is the introduction of class $m-cv \bigcap L^1_{loc}$, i.e. the definition of functions $m-cv(D)$ in the class of upper semi-continuous, locally integrable or bounded functions. So, for $m=n$ (the case of subharmonic functions) in the class of upper semi-continuous, locally integrable function $u(x) \in n-cv(D)$ is defined as a generalized function, and the Laplace operator Δu is a Borel measure.

2.3 Definition of Hessians for $m-cv$ Functions

In this work, we establish a connection between $m-cv$ functions and violent subharmonic (sh_m) functions and using the well-known and rich properties of sh_m functions we give the definitions of Hessians $H_k(u),\ \ k=1,\dots,n-m+1$ for $m-$convex functions, like Borel measures.
To do this, we embed $\mathbb{R}^n_x$ in $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y (z=x+iy)$, as a real $n-$dimensional subspace of the complex space $\mathbb{C}^n_z$.

Theorem 2.1 *A twice smooth function $u(x) \in C^2(D),\ \ D \subset \mathbb{R}^n_x$, is $m-cv$ in D if and only if the function $u^c(z)=u^c(x+iy)=u(x)$, that does not depend on variable $y \in \mathbb{R}^n_y$, is sh_m in the domain $D \times \mathbb{R}^n_y$.*

Proof Let us establish a connection between the Hessians $H_k(u)$ and $H^k(u^c)$. We have

$$\frac{\partial u^c}{\partial z_j} = \frac{1}{2}\left[\frac{\partial u^c}{\partial x_j} - \frac{\partial u^c}{\partial y_j}\right] = \frac{1}{2}\frac{\partial u^c}{\partial x_j},$$

$$\frac{\partial^2 u^c}{\partial z_j \bar{\partial}\, z_k} = \frac{1}{2}\frac{\partial}{\bar{\partial}\, z_k}\left[\frac{\partial u^c}{\partial x_j}\right] = \frac{1}{4}\left[\frac{\partial^2 u^c}{\partial x_k \partial x_j} + \frac{\partial^2 u^c}{\partial x_k \partial y_j}\right] = \frac{1}{4}\frac{\partial^2 u^c}{\partial x_k \partial x_j}.$$

Thus,

$$\frac{\partial^2 u^c}{\partial z_j \partial \bar{z}_k} = \frac{1}{4}\frac{\partial^2 u}{\partial x_j \partial x_k}$$

and, therefore, $H_k(u)=H^k(u^c)$ and $H^k(u) \geq 0,\ \ k=1,\dots,n-m+1$, if and only if $H^k(u^c) \geq 0,\ \ k=1,\dots,n-m+1$. □

Now, let $u(x)$ be an upper semi-continuous function in the domain $D \subset \mathbb{R}^n_x$. Then, $u^c(z)$ will also be an upper semi-continuous function in the domain $D \times \mathbb{R}^n_y \subset \mathbb{C}^n_z$.

Definition 2.3 An upper semi-continuous function $u(x)$ in a domain $D \subset \mathbb{R}_x^n$ is called $m-$convex if the function $u^c(z)$ is violent m-subharmonic, $u^c(z) \in sh_m(D \times \mathbb{R}_y^n)$.

If a function $u(x)$ is bounded and $m-$convex in the domain $D \subset \mathbb{R}_x^n$, then $u^c(z)$ will be a bounded function that is violent $m-$subharmonic in the domain $D \times \mathbb{R}_y^n \subset \mathbb{C}_z^n$. Therefore, the operators $(dd^c u^c)^k \wedge \beta^{n-k}, \ \ k = 1, 2, \ldots, n-m+1$ are defined like Borel measures in the domain $D \times \mathbb{R}_y^n \subset \mathbb{C}_z^n, \ \ \mu_k = (dd^c u^c)^k \wedge \beta^{n-k}$.

Since for a doubly smooth function $(dd^c u^c)^k \wedge \beta^{n-k} = k!(n-k)! H^k(u^c)\beta^n$, then for a bounded, violent $m-$subharmonic function in the domain $D \times \mathbb{R}_y^n \subset \mathbb{C}_z^n$, it is natural to determine its Hessians, equating them to the measure

$$H^k(u^c) = \frac{\mu_k}{k!(n-k)!} = \frac{1}{k!(n-k)!}(dd^c u^c)^k \wedge \beta^{n-k}. \tag{2.4}$$

Using (2.4), we can now define Hessians $H^k, \ \ k = 1, 2, \ldots, n-m+1$, in the class of bounded, $m-$convex functions in the domain $D \subset \mathbb{R}_x^n$.

Let $u(x)$ be a bounded function, $m-$convex in the domain $D \subset \mathrm{R}_x^n$. Let us define Borel measures $\mu_k = (dd^c u^c)^k \wedge \beta^{n-k}, \ \ k = 1, 2, \ldots, n-m+1$ in the domain $D \times \mathbb{R}_y^n \subset \mathbb{C}_z^n$.

Since $u^c \in sh_m(D \times \mathbb{R}_y^n)$ does not depend on $y \in \mathbb{R}_y^n$, for any Borel sets $E_x \subset D, \ \ E_y \subset \mathrm{R}_y^n$, the measures $\frac{1}{mes E_y}\mu_k(E_x \times E_y)$ do not depend on the set $E_y \subset \mathbb{R}_y^n$, i.e. $\frac{1}{mes E_y}\mu_k(E_x \times E_y) = \nu_k(E_x)$. We will call Borel measures ν_k : $\nu_k(E_x) = \frac{1}{mes E_y}\mu_k(E_x \times E_y)$, $k = 1, 2, \ldots, n-m+1$, the Hessians $H_k, k = 1, 2, \ldots, n-m+1$, for a bounded, $m-$convex function $u(x) \in m - cv(D)$ in the domain $D \subset \mathbb{R}_x^n$. For a doubly smooth function $u(x) \in m - cv(D) \bigcap C^2(D)$, the Hessians are ordinary functions; however, for a non-doubly smooth but bounded semi-continuous function $u(x) \in m - cv(D) \bigcap L^\infty(D)$, the Hessians $H_k, \ \ k = 1, 2, \ldots, n-m+1$ are positive Borel measures.

From Theorem 2.1 and (2.4) the following result follows easily.

Theorem 2.2 *A twice smooth function $u(x) \in C^2(D), \ x \in D \subset \mathbb{R}_x^n$ is $m - cv(D)$ if and only if*

$$\begin{aligned} &dd^c u^c \wedge dd^c v_1^c \wedge \ldots \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \geq 0, \\ &\forall v_1, \ldots, v_{n-m} \in m - cv(D) \textstyle\bigcap C^2(D). \end{aligned} \tag{2.5}$$

Note that Theorem 2.2 allows us to give a criterion for $u(x) \in m - cv(D)$ in the class $L_{loc}^1(D)$.

Definition 2.4 A function $u(x) \in L^1_{loc}(D)$ is called $m-$convex function in a domain $D \subset \mathbb{R}^n_x$, $u(x) \in m - cv(D)$, if it is upper semi-continuous and for any doubly smooth $m - cv(D)$ functions of $v_1, \ldots, v_{n-m}$ the following

$$\begin{aligned}&\left[dd^c u^c \wedge dd^c v_1^c \wedge \ldots \wedge dd^c v_{n-m}^c \wedge \beta^{m-1}\right](\omega) = \\ &= \int u^c dd^c v_1^c \wedge \ldots \wedge dd^c v_{n-m}^c \wedge \beta^{m-1} \wedge dd^c \omega, \qquad \omega \in F^{0,0}(D \times \mathbb{R}^n_y)\end{aligned} \tag{2.6}$$

is positive.

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References

1. Blocki, Z.: The domain of definition of the complex Monge-Ampere operator. Am. J. Math. **128**(2), 519–530 (2006)
2. Dinev, S., Kolodziej, S.: A priori estimates for the complex Hessian equation. Anal. PDE **7**, 227–244 (2014)
3. Li, S.Y.: On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian. Asian J. Math. **8**, 87–106 (2004)
4. Lu, H.C.: Solutions to degenerate Hessian equations. J. Math. Pures Appl. **100**(6), 785–805 (2013)
5. Lu, H.C., Nguyen, V.D.: Degenerate complex Hessian equations on compact Kahler manifolds. arXivmath:1402.5147
6. Abdullaev, B.I., Sadullaev, A.: Potential theory in the class of $m-$subharmonic functions (in Russian). Proc. V.A. Math. Inst. Steklova Moscow **279**, 166–192 (2012)
7. Abdullaev, B.I., Sadullaev, A.: Capacities and Hessians in the class of $m-$subharmonic functions (in Russian). Rep. Acad. Sci. Russia. **448**(5), 1–3 (2013)
8. Aleksandrov, A.D.: Intrinsic Geometry of Convex Surfaces. OGIZ, Moscow (1948); German transl., AkademieVerlag, Berlin (1955)
9. Aleksandrov, A.D.: Dirichlet's problem for the equation $\det(z_{ij}) = \varphi$, Vestnic Leningrad Univ. **13**, 5–24 (1958)
10. Bakelman, I.J.: Variational problems and elliptic Monge-Ampere equations. J. Differ. Geo **18**, 669–999 (1983)
11. Bakelman, I.J.: Complex Analysis and Nonlinear Geom. Ellip. Equat. Springer, New York (1994)
12. Pogorelov, A. V.: Extrinsic Geometry of Convex Surfaces. Nauka, Moscow (1969); English transl., Amer. Math. Soc, Providence (1973)
13. Caffarelly, L., Nirenberg, L., Spruck J.: Functions of the eigenvalues of the Hessian. Acta Math. **155**, 261–301 (1985)
14. Trudinger, N.S., Wang X.J.: Hessian measures I. Topol. Methods Nonlinear Anal. **10**, 225–239 (1997)
15. Trudinger, N.S., Wang, X.J.: Hessian measures II. Anal. Math. **150**, 579–604 (1999)
16. Trudinger, N.S., Wang, X.J.: Hessian measures III. J. Funct. Anal. **193**, 1–23 (2002)
17. Chou, K.S., Wang, X.J.: Variational theoryfor Hessian equations. Commun. Pure Appl. Math. **5**, 1029–1064 (2001)
18. Ivochkina, N., Trudinger, N.S., Wang X.J.: The Dirichlet problem for degenerate Hessian equations. Commun. Partial Diff. Equ. **29**, 219–235 (2004)

19. Wang, X.J.: The $k-$Hessian Equations. Lecture Notes in Mathematics, vol. 1977 (2009)
20. Artykbaev, A.: Recovering convex surfaces from the extrinsic curvature in Galilean space. Math. USSR-Sb. **47**(1), 195–214 (1984)
21. Bakelman, I.: Geometric Methods for Solving Elliptic Equations, pp 178–307. Nauka, Moscow (1965)

Chapter 3
Necessary and Sufficient Conditions for Basis Properties of the System of Root Functions of Sturm-Liouville Boundary Value Problems with Eigenparameter Dependent Boundary Conditions

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Abstract In this study, Sturm-Liouville problems with a boundary condition depending quadratically on an eigenparameter are considered. The necessary and sufficient conditions for minimality and completeness of the chosen system of root functions of the corresponding operator are given in two forms, one of which can be done by direct computations. This computationally simpler way was discussed in the literature for the affine case. But for the quadratic case it was done for a similar boundary value problem only recently. The aim of the present paper is to fill this gap for the quadratic case.

3.1 Introduction

In this paper, we study Sturm-Liouville problems with a boundary condition depending quadratically on an eigenparameter

$$-y'' + q(x)y = \lambda y,\ \ 0 < x < 1, \tag{3.1}$$

$$y(0)\cos\beta = y'(0)\sin\beta,\ \ 0 \le \beta < \pi, \tag{3.2}$$

$$y(1) = (a\lambda^2 + b\lambda + c)y'(1), \tag{3.3}$$

where λ is the spectral parameter, $q(x)$ is a continuous and real-valued function on the interval [0, 1], and a, b, c are real numbers.

We will study the quadratic case when $a \neq 0$. Affine case $(a = 0, b < 0)$ was studied in [1]. In analogy with [1, 2, 4, 5, 7] it can be shown that the eigenvalues λ_n

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of the problem (3.1)–(3.3) tend only to positive infinity and the following cases can take place: **(a)** all $\lambda_n \in \mathbb{R}$ and simple; **(b)** all $\lambda_n \in \mathbb{R}$ and all, except $\lambda_k = \lambda_{k+1}$, are simple; **(c)** all $\lambda_n \in \mathbb{R}$ and all, except $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$, are simple; **(d)** all are simple and all, except $\lambda_s = \overline{\lambda_r}$, are real. The eigenvalues λ_n $(n \geq 0)$ are listed with respect to non-decreasing real parts and repeated if there are multiple eigenvalues. Asymptotic formula for the eigenvalues λ_n and oscillations of eigenfunctions of the boundary value problem (3.1)–(3.3), with more general rational function of λ in (3.3) were discussed in [3].

The necessary and sufficient conditions for minimality and completeness of the chosen system of root functions of the operator were given in two forms, one of which is more direct. This simpler way was known for the affine case [1]. The present paper will extend these results for the quadratic case using the methods of [2, 5]. In more general and abstract setting these problems were studied in [8–10].

3.2 Inner Products and Norms of Root Functions

We denote by $y(x, \lambda)$ a nontrivial solution of (3.1), for initial conditions $y(0) = \sin\beta$ and $y'(0) = \sin\beta$. Then the characteristic equation is

$$\omega(\lambda) = y(1, \lambda) - (a\lambda^2 + b\lambda + c)y'(1, \lambda). \tag{3.4}$$

If $\omega(\lambda_n) = 0$ then by (3.3), λ_n is an eigenvalue of (3.1)–(3.3). If $\omega(\lambda_n) = 0$ and $\omega'(\lambda_n) \neq 0$, then λ_n is a simple eigenvalue. If $\omega(\lambda_k) = \omega'(\lambda_k) = 0$ and $\omega''(\lambda_k) \neq 0$, then $\lambda_k = \lambda_{k+1}$ is a double eigenvalue. If $\omega(\lambda_k) = \omega'(\lambda_k) = \omega''(\lambda_k) = 0$ and $\omega'''(\lambda_k) \neq 0$, then $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$ is a triple eigenvalue. As usual $(\cdot, \cdot)$ and $\|\cdot\|$ are inner product and norm in $L_2(0, 1)$, respectively.

Lemma 3.1 *If y_n, y_m are eigenfunctions with eigenvalues λ_n, λ_m $(\lambda_n \neq \overline{\lambda_m})$, respectively, then*

$$(y_n, y_m) = (a\lambda_n + b + a\overline{\lambda_m})y'_n(1)\overline{y'_m(1)}.$$

Lemma 3.2 *If $\lambda_n \in \mathbb{R}$ then*

$$\|y_n\|_2^2 = (2a\lambda_n + b)y'_n(1)^2 + y'_n(1)\omega'(\lambda_n).$$

Corollary 3.1 *If eigenvalue $\lambda_k = \lambda_{k+1}$ is double or triple then*

$$\|y_k\|_2^2 = (2a\lambda_k + b)y'_k(1)^2.$$

Corollary 3.2 *If $\lambda_r \notin \mathbb{R}$, then*

$$\|y_r\|_2^2 = (2a\mathrm{Re}\lambda_r + b)|y'_r(1)|^2.$$

Definition 3.1 If y_n is an eigenfunction then let

$$B_n = \|y_n\|_2^2 + (2a\mathrm{Re}\lambda_n + b)|y_n'(1)|^2.$$

Corollary 3.3 *Eigenvalue λ_n is real and simple if and only if $B_n = 0$.*

If $\lambda_k = \lambda_{k+1}$ then let $B_{k+1} = y_k'(1)\omega''(\lambda_k)/2$. If $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$ then let $B_{k+2} = y_k'(1)\omega'''(\lambda_k)/6$.

Lemma 3.3 *If $\lambda_r, \lambda_s \notin \mathbb{R}$ and $\lambda_s = \overline{\lambda_r}$, then*

$$(y_r, y_s) = (2a\lambda_r + b)y_r'(1)^2 + y_r'(1)\omega'(\lambda_r).$$

Definition 3.2 If $\lambda_k = \lambda_{k+1}$, then the 1st associated function y_{k+1} is defined by

$$-y_{k+1}'' + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k,$$

$$y_{k+1}(0)\cos\beta = y_{k+1}'(0)\sin\beta,$$

$$y_{k+1}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+1}'(1) + (2a\lambda_k + b)y_k'(1).$$

Definition 3.3 If $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$ then the 2nd associated function y_{k+2} is defined by

$$-y_{k+2}'' + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1},$$

$$y_{k+2}(0)\cos\beta = y_{k+2}'(0)\sin\beta,$$

$$y_{k+2}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+2}'(1) + (2a\lambda_k + b)y_{k+1}'(1) + ay_k'(1).$$

It is well known that associated functions are not unique in the sense that $y_{k+1} + Ay_k$ and $y_{k+2} + By_k$, for constants A and B, are also first and second associated functions, respectively. It is also not difficult to check that the change of y_{k+1} to $y_{k+1} + Ay_k$, changes y_{k+2} to $y_{k+2} + Ay_{k+1}$. If $\lambda_k = \lambda_{k+1}$ $(= \lambda_{k+2})$ then for $y(x,\lambda)$ and its derivatives with respect to λ we can write $\lim_{\lambda\to\lambda_k} y(x,\lambda) = y_k$, $\lim_{\lambda\to\lambda_k} y_\lambda(x,\lambda) = \tilde{y}_{k+1}$, $\lambda_k = \lambda_{k+1}$ $(\lim_{\lambda\to\lambda_k} y_{\lambda\lambda} = 2\tilde{y}_{k+2})$, all uniformly, where $\tilde{y}_{k+1} = y_{k+1} + \tilde{C}y_k$, $\tilde{y}_{k+2} = y_{k+2} + \tilde{C}y_{k+1} + \tilde{D}y_k$, for some constants $\tilde{C}$ and $\tilde{D}$.

Lemma 3.4 *If $\lambda_n \neq \lambda_k = \lambda_{k+1}$, then*

$$(y_{k+1}, y_n) = (a\lambda_k + b + a\lambda_n)y_{k+1}'(1)y_n'(1) + ay_k'(1)y_n'(1),$$

$$(y_{k+1}, y_k) = (2a\lambda_k + b)y_{k+1}'(1)y_k'(1) + a(y_k'(1))^2 + B_{k+1},$$

$$\|y_{k+1}\|_2^2 = (2a\lambda_k + b)(y'_{k+1}(1))^2 + 2ay'_{k+1}(1)y'_k(1) + \widehat{y}'_{k+1}(1)\frac{\omega''(\lambda_k)}{2} + B_{k+2},$$

where $\widehat{y}_{k+1} = y_{k+1} - \tilde{C}y_k$.

Lemma 3.5 *If* $\lambda_n \neq \lambda_k = \lambda_{k+1} = \lambda_{k+2}$, *then*

$$(y_{k+2}, y_n) = (a\lambda_k + b + a\lambda_n)y'_{k+2}(1)y'_n(1) + ay'_{k+1}(1)y'_n(1),$$

$$(y_{k+2}, y_k) = (2a\lambda_k + b)y'_{k+2}(1)y'_k(1) + ay'_{k+1}(1)y'_k(1) + B_{k+2},$$

$$(y_{k+2}, y_{k+1}) = (2a\lambda_k + b)y'_{k+2}(1)y'_{k+1}(1) + ay'_{k+2}(1)y'_k(1) + a(y'_{k+1}(1))^2 + \widehat{y}'_{k+1}(1)\frac{\omega'''(\lambda_k)}{6} + y'_k(1)\frac{\omega^{IV}(\lambda_k)}{24},$$

$$\|y_{k+2}\|_2^2 = (2a\lambda_k + b)y'_{k+2}(1)^2 + 2ay'_{k+2}(1)y'_{k+1}(1) + \widehat{y}'_{k+2}(1)\frac{\omega'''(\lambda_k)}{6} + \widehat{y}'_{k+1}(1)\frac{\omega^{IV}(\lambda_k)}{24} + y'_k(1)\frac{\omega^{V}(\lambda_k)}{120},$$

where $\widehat{y}_{k+2} = y_{k+2} - \tilde{C}\widehat{y}_{k+1} - \tilde{D}y_k$.

Lemma 3.6 *If* $\lambda_k = \lambda_{k+1} \neq \lambda_{k+2}$ *then there is a constant* C_1, *such that* $y^*_{k+1} = y_{k+1} + C_1 y_k$ *satisfies*

$$(y^*_{k+1}, y_{k+1}) = (2a\lambda_k + b)(y^*_{k+1})'(1)y'_{k+1}(1) + a(y^*_{k+1})'(1)y'_k(1) + ay'_{k+1}(1)y'_k(1).$$

Remark 3.1 Note that

$$C_1 = -\frac{2B_{k+2} + \widehat{y}'_{k+1}(1)\omega''(\lambda_k)}{2B_{k+1}}.$$

Lemma 3.7 *If* $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$ *then there are constants* C_2 *and* D_1 *such that* $y^\#_{k+1} = y_{k+1} + C_2 y_k$ *and* $y^\#_{k+2} = y_{k+2} + C_2 y_{k+1} + D_1 y_k$ *satisfy*

$$(y^\#_{k+1}, y_{k+2}) = (2a\lambda_k + b)(y^\#_{k+1})'(1)y'_{k+2}(1) + a(y^\#_{k+1})'(1)y'_{k+1}(1) + ay'_{k+2}(1)y'_k(1),$$

$$(y^\#_{k+2}, y_{k+1}) = (2a\lambda_k + b)(y^\#_{k+2})'(1)y'_{k+1}(1) + a(y^\#_{k+2})'(1)y'_k(1) + a(y^\#_{k+1})'(1)y'_{k+1}(1),$$

$$(y^\#_{k+2}, y_{k+2}) = (2a\lambda_k + b)(y^\#_{k+2})'(1)y'_{k+2}(1) + a(y^\#_{k+2})'(1)y'_{k+1}(1) + a(y^\#_{k+1})'(1)y'_{k+2}(1).$$

Remark 3.2 Note that

$$C_2 = -\frac{\widehat{y}'_{k+1}(1)\frac{\omega'''(\lambda_k)}{3!} + y'_k(1)\frac{\omega^{IV}(\lambda_k)}{4!}}{B_{k+2}},$$

$$D_1 = -\frac{\widehat{y}'_{k+2}(1)\frac{\omega'''(\lambda_k)}{3!} + \widehat{y}'_{k+1}(1)\frac{\omega^{IV}(\lambda_k)}{4!} + y'_k(1)\frac{\omega^{V}(\lambda_k)}{5!}}{B_{k+2}} + C_2^2.$$

3.3 Equivalent Minimality Conditions

Cases **a** and **d**.

Theorem 3.1 *If all λ_n are simple (cases **a** and **d**), then the system $\{y_n\}$ $(n \in \mathbb{Z}_{\geq 0};\ n \neq i, j)$, where $i, j \in \mathbb{Z}_{\geq 0}$ and $i \neq j$, is minimal in $L_2(0, 1)$.*

Proof In case **a**, it is sufficient to show that there is a biorthogonal system

$$\{u_n\}\ (n \in \mathbb{Z}_{\geq 0};\ n \neq i, j),$$

elements of which can be taken as

$$u_n(x) = \frac{1}{B_n \Delta_{ij}} \begin{vmatrix} y_n(x) & y'_n(1) & \lambda_n y'_n(1) \\ y_i(x) & y'_i(1) & \lambda_i y'_i(1) \\ y_j(x) & y'_j(1) & \lambda_j y'_j(1) \end{vmatrix},$$

where $\Delta_{ij} = (\lambda_j - \lambda_i) y_i(1) y_j(1)$. Verification of $(u_n, y_m) = \delta_{nm}$ $(n, m \neq i, j)$, where δ_{nm} is Kronecker's symbol: $\delta_{nm} = 0$ if $n \neq m$ and $\delta_{nn} = 1$, is done using the results of the previous section.

In case **d**, if $\lambda_s = \overline{\lambda_r}$, then for the cases $i, j \neq r, s$; $i = r, j \neq s$; $i \neq r, j = s$; $i = r, j = s$ it is necessary to change the biortogonal system accordingly (see [2]). □

Case **b**.

Theorem 3.2 *If $\lambda_k = \lambda_{k+1} \neq \lambda_{k+2}$, then*

$$\{y_n\}\ (n \in \mathbb{Z}_{\geq 0};\ n \neq k, k+1),$$

$$\{y_n\}\ (n \in \mathbb{Z}_{\geq 0};\ n \neq k+1, j),\ j \in \mathbb{Z}_{\geq 0},\ j \neq k, k+1$$

$$\{y_n\}\ (n \in \mathbb{Z}_{\geq 0};\ n \neq i, j),\ i, j \in \mathbb{Z}_{\geq 0},\ i, j \neq k, k+1,$$

are minimal systems in $L_2(0, 1)$.

Proof In the first case the biorthogonal system is defined for $n \neq k, k+1$ by

$$u_n(x) = \frac{1}{B_n y_k'(1)^2} \begin{vmatrix} y_n(x) & y_n'(1) & \lambda_n y_n'(1) \\ y_k(x) & y_k'(1) & \lambda_k y_k'(1) \\ y_{k+1}(x) & y_{k+1}'(1) & \lambda_k y_{k+1}'(1) + y_k'(1) \end{vmatrix}. \tag{3.5}$$

In the remaining cases, it is necessary to change the biorthogonal system accordingly (see [2]). □

Theorem 3.3 *If* $\lambda_k = \lambda_{k+1} \neq \lambda_{k+2}$ *then*

$$\{y_n\} \ (n \in \mathbb{Z}_{\geq 0};\ n \neq k, j),\ j \in \mathbb{Z}_{\geq 0},\ j \neq k, k+1$$

is a minimal system in $L_2(0, 1)$ *if and only if* $(y_{k+1}^*)'(1)(\lambda_j - \lambda_k) \neq y_k'(1)$.

Proof We define the biorthogonal system as

$$u_n(x) = \frac{1}{B_n \Delta_{kj}^*} \begin{vmatrix} y_n(x) & y_n'(1) & \lambda_n y_n'(1) \\ y_{k+1}^*(x) & (y_{k+1}^*)'(1) & \lambda_k (y_{k+1}^*)'(1) + y_k'(1) \\ y_j(x) & y_j'(1) & \lambda_j y_j'(1) \end{vmatrix},$$

for $n \neq k, k+1, j$, and

$$u_{k+1}(x) = \frac{1}{B_{k+1} \Delta_{kj}^*} \begin{vmatrix} y_k(x) & y_k'(1) & \lambda_k y_k'(1) \\ y_{k+1}^*(x) & (y_{k+1}^*)'(1) & \lambda_k (y_{k+1}^*)'(1) + y_k'(1) \\ y_j(x) & y_j'(1) & \lambda_j y_j'(1) \end{vmatrix},$$

where $\Delta_{kj}^* = (\lambda_j - \lambda_k)(y_{k+1}^*)'(1)' y_j(1) - y_k'(1) y_j'(1)$. If $\Delta_{kj}^* = 0$ then

$$g_1(x) = \begin{vmatrix} y_k(x) & y_k'(1) & \lambda_k y_k'(1) \\ y_{k+1}(x) & y_{k+1}'(1) & \lambda_k y_{k+1}'(1) + y_k'(1) \\ y_j(x) & y_j'(1) & \lambda_j y_j'(1) \end{vmatrix}$$

is orthogonal to all the elements of the system $\{y_n\}$ $(n \neq k, j)$ and consequently it is not complete in $L_2(0, 1)$. It is also not minimal in $L_2(0, 1)$ because otherwise using its minimality in $L_2(0, 1)$, one can show that it is a basis of $L_2(0, 1)$, which contradicts with its incompleteness in $L_2(0, 1)$ (see [2]). □

Remark 3.3 Note that $(y_{k+1}^*)'(1)(\lambda_j - \lambda_k) \neq y_k'(1)$ if and only if (see [5])

$$\widetilde{C} \neq \frac{1}{\lambda_j - \lambda_k} + \frac{1}{3} \cdot \frac{\omega'''(\lambda_k)}{\omega''(\lambda_k)}.$$

Case **c**.

Theorem 3.4 *If* $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$, *then*

$$\{y_n\}\ \ (n \in \mathbb{Z}_{\geq 0};\ \ n \neq k+1, k+2),$$

$$\{y_n\}\ \ (n \in \mathbb{Z}_{\geq 0};\ \ n \neq k+2, j),\ j \in \mathbb{Z}_{\geq 0},\ j \neq k, k+1, k+2,$$

$$\{y_n\}\ \ (n \in \mathbb{Z}_{\geq 0};\ \ n \neq i, j),\ i, j \in \mathbb{Z}_{\geq 0},\ i, j \neq k, k+1, k+2,$$

are minimal systems in $L_2(0, 1)$.

Theorem 3.5 *If* $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$, *then*

$$\{y_n\}\ \ (n \in \mathbb{Z}_{\geq 0};\ \ n \neq k, k+2),$$

is a minimal system in $L_2(0, 1)$ *if and only if* $(y^{\#}_{k+1})'(1) \neq 0$.

Remark 3.4 Note that $(y^{\#}_{k+1})'(1) \neq 0$ if and only if (see [5])

$$\widetilde{C} \neq \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}.$$

Also note that if $(y^{\#}_{k+1})'(1) = 0$, then

$$g_2(x) = \begin{vmatrix} y_k(x) & y'_k(1) & \lambda_k y'_k(1) \\ y_{k+1}(x) & y'_{k+1}(1) & \lambda_k y'_{k+1}(1) + y'_k(1) \\ y_{k+2}(x) & y'_{k+2}(1) & \lambda_k y'_{k+2}(1) + y'_{k+1}(1) \end{vmatrix},$$

is orthogonal to all the elements of the system $\{y_n\}\ \ (n \in \mathbb{Z}_{\geq 0};\ \ n \neq k, k+2)$.

Theorem 3.6 *If* $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$, *then* $\{y_n\}\ \ (n \in \mathbb{Z}_{\geq 0};\ \ n \neq k, k+1)$ *is a minimal system in* $L_2(0, 1)$ *if and only if* $((y^{\#}_{k+1})'(1))^2 \neq y'_k(1)(y^{\#}_{k+2})'(1)$.

Remark 3.5 Note that $((y^{\#}_{k+1})'(1))^2 \neq y'_k(1)(y^{\#}_{k+2})'(1)$ if and only if

$$\widetilde{D} \neq \widetilde{C} \cdot \left(\widetilde{C} - \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) + \frac{1}{20} \cdot \frac{\omega^{V}(\lambda_k)}{\omega'''(\lambda_k)}.$$

Otherwise, $g_2(x)$ is orthogonal to all the elements of the last system.

Theorem 3.7 *If* $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$ *then*

$$\{y_n\}\ \ (n \in \mathbb{Z}_{\geq 0};\ \ n \neq k+1, j),\ j \in \mathbb{Z}_{\geq 0},\ j \neq k, k+1, k+2,$$

is a minimal system in $L_2(0, 1)$ *if and only if* $(y^{\#}_{k+1})'(1)(\lambda_j - \lambda_k) \neq y'_k(1)$.

Remark 3.6 Note that $(y_{k+1}^{\#})'(1)(\lambda_j - \lambda_k) \neq y_k'(1)$ if and only if

$$\widetilde{C} \neq \frac{1}{\lambda_j - \lambda_k} + \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}.$$

Note also that if $(y_{k+1}^{\#})'(1)(\lambda_j - \lambda_k) = y_k'(1)$, then

$$g_3(x) = \begin{vmatrix} y_k(x) & y_k'(1) & \lambda_k y_k'(1) \\ y_{k+1}(x) & y_{k+1}'(1) & \lambda_k y_{k+1}'(1) + y_k'(1) \\ y_j(x) & y_j'(1) & \lambda_j y_j'(1) \end{vmatrix},$$

is orthogonal to all the elements of the system $\{y_n\}$ $(n \in \mathbb{Z}_{\geq 0};\ n \neq k+1, j)$.

Theorem 3.8 *If* $\lambda_k = \lambda_{k+1} = \lambda_{k+2}$, *then*

$$\{y_n\}\ (n \in \mathbb{Z}_{\geq 0};\ n \neq k, j),\ j \in \mathbb{Z}_{\geq 0},\ j \neq k, k+1, k+2,$$

is a minimal system in $L_2(0, 1)$ *if and only if* $(y_{k+2}^{\#})'(1)(\lambda_j - \lambda_k) \neq (y_{k+1}^{\#})'(1)$.

Remark 3.7 Note that $(y_{k+2}^{\#})'(1)(\lambda_j - \lambda_k) \neq (y_{k+1}^{\#})'(1)$ if and only if

$$\widetilde{D} \neq \left(\frac{1}{\lambda_j - \lambda_k} + \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) \cdot \left(\widetilde{C} - \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) + \frac{1}{20} \cdot \frac{\omega^{V}(\lambda_k)}{\omega'''(\lambda_k)}.$$

Note also that if $(y_{k+2}^{\#})'(1)(\lambda_j - \lambda_k) \neq (y_{k+1}^{\#})'(1)$ then

$$g_4(x) = \begin{vmatrix} y_k(x) & y_k'(1) & \lambda_k y_k'(1) \\ y_{k+2}^{\#}(x) & (y_{k+2}^{\#})'(1) & \lambda_k (y_{k+2}^{\#})'(1) + (y_{k+1}^{\#})'(1) \\ y_j(x) & y_j'(1) & \lambda_j y_j'(1) \end{vmatrix},$$

is orthogonal to all the elements of the system $\{y_n\}$ $(n \in \mathbb{Z}_{\geq 0};\ n \neq k, j)$.

All the minimality results in this section can be generalized to basis properties in $L_p(0, 1)$ $(1 < p < \infty)$ using the method of [2].

3.4 Example

We are going to apply the obtained results to the following example:

$$-y'' = \lambda y,\ 0 < x < 1,$$

$$y(0) = 0,\ y(1) = \left(\frac{2\lambda^2}{15} + \frac{\lambda}{3} + 1\right) y'(1).$$

For this problem $y(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$ and

$$\omega(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - \left(\frac{2\lambda^2}{15} + \frac{\lambda}{3} + 1\right)\cos\sqrt{\lambda}.$$

We calculate the limits

$$\omega(0) = \omega'(0) = \omega''(0) = 0,\ \omega'''(0) = \frac{34}{105},\ \omega^{IV}(0) = -\frac{116}{945},\ \omega^{V}(0) = \frac{221}{10395}.$$

So, $\lambda_0 = \lambda_1 = \lambda_2 = 0$ is a triple eigenvalue and the remaining eigenvalues $\lambda_n > 0$ $(n = 3, 4, \ldots)$ can be determined from $\omega(\lambda) = 0$, which can also be written as

$$\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}} = \frac{2\lambda^2}{15} + \frac{\lambda}{3} + 1.$$

Note that $\lambda_3 \approx 22.179568975$. For numerical methods of approximation of eigenvalues for such problems see e.g. [6]. The corresponding eigenfunctions are $y_0 = x$, $y_n = \sin\sqrt{\lambda_n}x$ $(n \geq 3)$ and associated functions are $y_1 = -\frac{1}{6}x^3 + Cx$, $y_2 = \frac{1}{120}x^5 + C\left(-\frac{1}{6}x^3 + Cx\right) + Dx$, where C and D are constants. We also find limits $\tilde{y}_1 = -\frac{1}{6}x^3$, $\tilde{y}_2 = \frac{1}{120}x^5$, which means that $\tilde{C} = -C$ and $\tilde{D} = -D$.

By Theorem 3.5, the system

$$\{y_n\}\ (n \in \mathbb{Z}_{\geq 0};\ n \neq 0, 2) = \left\{-\frac{1}{6}x^3 + Cx, \sin\sqrt{\lambda_n}x\ (n \geq 3)\right\}$$

is minimal in $L_2(0,1)$ if and only if

$$\widetilde{C} \neq \frac{1}{4} \cdot \frac{\omega^{IV}(0)}{\omega'''(0)} = -\frac{29}{306},$$

which can be written as $C \neq \frac{29}{306}$. If $C = \frac{29}{306}$, then

$$g_2(x) = \begin{vmatrix} y_0 & y_0'(1) & \lambda_0 y_0'(1) \\ y_1 & y_1'(1) & \lambda_0 y_1'(1) + y_0'(1) \\ y_2 & y_2'(1) & \lambda_0 y_2'(1) + y_1'(1) \end{vmatrix} = \frac{1}{120}x^5 - \frac{1}{12}x^3 + \frac{5}{24}$$

is orthogonal to all the elements of the system $\{y_n\}$ $(n \in \mathbb{Z}_{\geq 0};\ n \neq 0, 2)$.

By Theorem 3.6, the system

$$\{y_n\}\ (n \in \mathbb{Z}_{\geq 0};\ n \neq 0, 1)$$
$$= \left\{\frac{1}{120}x^5 + C\left(-\frac{1}{6}x^3 + Cx\right) + Dx, \sin\sqrt{\lambda_n}x\ (n \geq 3)\right\}$$

is minimal in $L_2(0, 1)$ if and only if

$$\widetilde{D} \neq \widetilde{C} \cdot \left(\widetilde{C} - \frac{1}{4} \cdot \frac{\omega^{IV}(0)}{\omega'''(0)} \right) + \frac{1}{20} \cdot \frac{\omega^{V}(0)}{\omega'''(0)},$$

which simplifies to $D \neq -C^2 + \frac{29}{306}C - \frac{13}{3960}$. If $D = -C^2 + \frac{29}{306}C - \frac{13}{3960}$ then the same function $g_2(x) = \frac{1}{120}x^5 - \frac{1}{12}x^3 + \frac{5}{24}$ is orthogonal to all the elements of the system $\{y_n\}$ $(n \in \mathbb{Z}_{\geq 0};\ n \neq 0, 1)$.

By Theorem 3.7 (take $j = 3$), the system $\{y_n\}$ $(n \in \mathbb{Z}_{\geq 0};\ n \neq 1, 3)$ i.e.

$$\left\{ x,\ \frac{1}{120}x^5 + C\left(-\frac{1}{6}x^3 + Cx\right) + Dx, \sin\sqrt{\lambda_n}x\ (n \geq 4) \right\}$$

is minimal in $L_2(0, 1)$ if and only if $\widetilde{C} \neq \frac{1}{\lambda_3} + \frac{1}{4} \cdot \frac{\omega^{IV}(0)}{\omega'''(0)}$, which is equivalent to $C \neq -\frac{1}{\lambda_3} + \frac{29}{306}$. If $C = -\frac{1}{\lambda_3} + \frac{29}{306}$, then

$$g_3(x) = \begin{vmatrix} y_0 & y_0'(1) & \lambda_0 y_0'(1) \\ y_1 & y_1'(1) & \lambda_0 y_1'(1) + y_0'(1) \\ y_3 & y_3'(1) & \lambda_3 y_3'(1) \end{vmatrix}$$

$$= \begin{vmatrix} x & 1 & 0 \\ -\frac{1}{6}x^3 + Cx & -\frac{1}{2} + C & 1 \\ \sin\sqrt{\lambda_3}x & \sqrt{\lambda_3}\sin\sqrt{\lambda_3} & \sqrt{\lambda_3^3}\sin\sqrt{\lambda_3} \end{vmatrix}$$

is orthogonal to all the elements of the system $\{y_n\}$ $(n \in \mathbb{Z}_{\geq 0};\ n \neq 1, 3)$.

By Theorem 3.8 (take again $j = 3$), the system $\{y_n\}$ $(n \in \mathbb{Z}_{\geq 0};\ n \neq 0, 3)$ i.e.

$$\left\{ -\frac{1}{6}x^3 + Cx,\ \frac{1}{120}x^5 + C\left(-\frac{1}{6}x^3 + Cx\right) + Dx, \sin\sqrt{\lambda_n}x\ (n \geq 4) \right\}$$

is minimal in $L_2(0, 1)$ if and only if

$$\widetilde{D} \neq \left(\frac{1}{\lambda_3} + \frac{1}{4} \cdot \frac{\omega^{IV}(0)}{\omega'''(0)} \right) \cdot \left(\widetilde{C} - \frac{1}{4} \cdot \frac{\omega^{IV}(0)}{\omega'''(0)} \right) + \frac{1}{20} \cdot \frac{\omega^{V}(0)}{\omega'''(0)},$$

which simplifies to $D \neq \left(\frac{1}{\lambda_3} - \frac{29}{306}\right) \cdot \left(C - \frac{29}{306}\right) - \frac{13}{3960}$. Otherwise, using $C_2 = -2C + \frac{91}{153}$ we find that

$$g_4(x) = \begin{vmatrix} y_0 & y_0'(1) & \lambda_0 y_0'(1) \\ y_2^{\#} & (y_2^{\#})'(1) & \lambda_0 (y_2^{\#})'(1) + (y_1^{\#})'(1) \\ y_3 & y_3'(1) & \lambda_3 y_3'(1) \end{vmatrix}$$

$$= \begin{vmatrix} x & 1 & 0 \\ y_2 + C_2 y_1 + D_1 y_0 & y_2'(1) + C_2 y_1'(1) + D_1 y_0(1) & y_1'(1) + C_2 y_0'(1) \\ \sin\sqrt{\lambda_3}x & \sqrt{\lambda_3}\sin\sqrt{\lambda_3} & \sqrt{\lambda_3^3}\sin\sqrt{\lambda_3} \end{vmatrix}$$

$$= \begin{vmatrix} x & 1 & 0 \\ \frac{x^5}{120} - (C + C_2)\frac{x^3}{6} & \frac{1}{24} - \frac{C+C_2}{2} & -\frac{1}{2} + C + C_2 \\ \sin\sqrt{\lambda_3}x & \sqrt{\lambda_3}\sin\sqrt{\lambda_3} & \sqrt{\lambda_3^3}\sin\sqrt{\lambda_3} \end{vmatrix}$$

is orthogonal to all the elements of the system $\{y_n\}$ $(n \in \mathbb{Z}_{\geq 0};\ n \neq 0, 3)$.

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References

1. Aliyev, Y.N.: Minimality properties of Sturm-Liouville problems with increasing affine boundary conditions. In: Bastos, M.A., Castro, L., Karlovich, A.Y. (eds.) Operator Theory, Functional Analysis and Applications. Operator Theory: Advances and Applications, vol. 282, pp. 33–49. Birkhäuser, Cham (2021)
2. Aliyev, Y.N., Kerimov, N.B.: The basis property of Sturm-Liouville problems with boundary conditions depending quadratically on the eigenparameter. Arabian J. Sci. Eng. **33**, 123–136 (2008)
3. Binding, P.A., Browne, P.J., Watson, B.A.: Equivalence of inverse Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter. J. Math. Anal. Appl. **291**, 246–261 (2004)
4. Code, W.J., Browne, P.J.: Sturm-Liouville problems with boundary conditions depending quadratically on the eigenparameter. J. Math. Anal. Appl. **309**, 729–742 (2005)
5. Kerimov, N., Aliyev, Y.: Minimality conditions for Sturm-Liouville problems with a boundary condition depending affinely or quadratically on an eigenparameter. In: Advances in Functional Analysis and Operator Theory. Contemporary Mathematics, vol. 798, pp. 1–12. American Mathematical Society, Providence (2024). https://doi.org/10.1090/conm/798/15978
6. Liu, C.S., Chen, Y.W., Chang, C.W.: Precise eigenvalues in the solutions of generalized Sturm-Liouville problems. Math. Comput. Simul. **217**, 354–373 (2024)
7. Maris, E.A., Goktas, S.: A study on the uniform convergence of spectral expansions for continuous functions on a Sturm-liouville problem. Miskolc Math. Not. **20**(2), 1063–1081 (2019)

8. Russakovskii, E.M.: Operator treatment of a boundary-value problem with the spectral parameter appearing rationally in the boundary conditions (in Russian). Theory Funct. Funct. Anal. Appl. **30**, 120–128 (1978)
9. Shkalikov, A.A.: Boundary value problems for ordinary differential equations with a parameter in the boundary conditions (in Russian). Tr. Semin. Im. I. G. Petrovskogo **9**, 190–229 (1983)
10. Shkalikov, A.A.: Basis properties of root fnctions of differential operators with spectral prameter in the boundary conditions. Differ. Equ. **55**(5), 631–643 (2019)

Chapter 4
Asymptotic Analysis of Sturm–Liouville Problem with Two-Point Boundary Conditions

Artūras Štikonas

Abstract We investigate the characteristic equation for Sturm–Liouville problem with one classical Robin type boundary condition and another two-point nonlocal boundary condition. Finally, we obtain asymptotic expansions for eigenvalues and eigenfunctions.

4.1 Introduction

Consider the following one-dimensional Sturm–Liouville equation

$$-u''(t)+q(t)u(t)=\lambda u(t),\quad t\in[0,1],\tag{4.1}$$

where the real-valued function $q\in C[0,1]$; $\lambda=s^2$ is a complex spectral parameter and $s=x+\imath y$; $x,y\in\mathbb{R}$. We will use the notation $q_0:=2\int_0^1|q(\tau)|\,d\tau$, $Q(t)=\frac{1}{2}\int_0^t q(\tau)d\tau$.

We shall investigate Sturm–Liouville Problem (SLP) which consists of Eq. (4.1) on [0, 1] with one classical (local) Robin type Boundary Condition (BC)

$$\cos\alpha\; u(0)+\sin\alpha\; u'(0)=0,\quad \alpha\in(0,\pi),\tag{4.2}$$

and another two-point Nonlocal Boundary Condition (NBC)

$$\begin{aligned}
&\text{(Case 1)} && u'(1)=\gamma u(\xi), && \xi\in[0,1],\\
&\text{(Case 2)} && u'(1)=\gamma u'(\xi), && \xi\in[0,1),\\
&\text{(Case 3)} && u(1)=\gamma u(\xi), && \xi\in[0,1),
\end{aligned}\tag{4.3}$$

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where $\gamma \in \mathbb{R}$. We consider the Dirichlet and the Neumann BC:

$$\begin{aligned} &\text{(Case d)} \quad u(0) = 0, \\ &\text{(Case n)} \quad u'(0) = 0, \end{aligned} \tag{4.4}$$

too. The Sturm–Liouville problem (4.1), (4.3) in Case 3, (4.4) in Case d was investigated in [2, 5], the Sturm–Liouville problem (4.1), (4.3), (4.4) in Case n was investigated in [4].

4.2 Asymptotic Expansions for Initial Value Problem

In this section we present some statements about solution of IVP. These statements were proved in [3]. We will use them for investigation of asymptotic expansions for SLP (4.1)–(4.3). Additionally, we introduce some notation related to our asymptotical analysis of this problem.

Let $\lambda = s^2$, $s \in \mathbb{C}_s$ and $\omega_{\alpha s}(t)$ be a solution of Eq. (4.1) satisfying the initial conditions

$$\omega_{\alpha s}(0) = \sin\alpha, \qquad \omega'_{\alpha s}(0) = -\cos\alpha. \tag{4.5}$$

The function $\omega(t, s, \alpha) = \omega_{\alpha s}(t)$ is an analytic (holomorphic) function of s and this function satisfies boudary condition (4.2). We denote $\varphi_s(t) := \omega_{0s}(t) = \omega(t, s, 0)$ and $\psi_s(t) := \omega_{\pi/2,s}(t) = \omega(t, s, \pi/2)$.

The following integral equation holds [1, 6]:

$$\omega_{\alpha s}(t) - \frac{1}{s}\int_0^t q(\tau)\sin\bigl(s(t-\tau)\bigr)\omega_{\alpha s}(\tau)d\tau = \sin\alpha\cos(st) - \cos\alpha\frac{\sin(st)}{s}. \tag{4.6}$$

Under the condition that $q \in C^r[0, 1]$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, asymptotic expansions may be obtained for $\varphi_s(t)$ [3] and $\psi_s(t)$ [4]. We will use recursive formula

$$p^0_{i+1}(t) = -\frac{1}{2}\int_0^t q(\tau)p^0_i(\tau)\,d\tau - \sum_{j=2+\varrho}^{i} \frac{(qp^0_{j-1})^{(i-j)}(t) + (-1)^i(qp^0_{j-1})^{(i-j)}(0)}{2^{i-j+2}}. \tag{4.7}$$

Lemma 4.1 (See [3, Lemma 7]) *Let $s \in \mathbb{C}_s$ and $q \in C^r[0, 1]$. Then for $|s| \geq q_0$ we have the asymptotic expansions*

$$(\varphi_s)^{(l)}_s(t, s) = -\sum_{j=1}^{r+1} p^l_j(t)\cos\Bigl(st + \frac{\pi}{2}(j - l)\Bigr)s^{-j} + O\bigl(s^{-(r+2)}e^{(r+2)|y|t}\bigr), \tag{4.8}$$

$$(\varphi_s')_s^{(l)}(t,s) = -\sum_{j=0}^{r} \bar{p}_j^{\,l}(t)\cos\Big(st + \frac{\pi}{2}(j-l)\Big)s^{-j} + O\big(s^{-(r+1)}\mathrm{e}^{(r+2)|y|t}\big) \tag{4.9}$$

for $l \in \mathbb{N}_0$, *where* $p_1^k(t) = -tp_1^{k-1}(t)$, $p_i^k(t) = (1-i)p_{i-1}^{k-1}(t) - tp_i^{k-1}(t)$, $i = \overline{2, r+1}$, $\bar{p}_0^{\,k}(t) = -t\bar{p}_0^{\,k-1}(t)$, $\bar{p}_i^{\,k}(t) = (1-i)\bar{p}_{i-1}^{\,k-1}(t) - t\bar{p}_i^{\,k-1}(t)$, $i = \overline{1, r}$, $k \in \mathbb{N}$, $\bar{p}_i^{\,0}(t) = p_i^{0\prime}(t) - p_{i+1}^0(t)$, $i = \overline{1, r}$, $\bar{p}_0^{\,0}(t) = 1$, *and* $p_j^0(t)$ *is calculated by (4.7) for* $i = \overline{1, r}$ *with* $p_1^0(t) = -1$ *and* $\varrho = 0$.

Proof Let $\omega_s(t)$ be a solution of Eq. (4.1) satisfying the initial conditions

$$\omega_s(0) = 0, \qquad \omega_s'(0) = -1. \tag{4.10}$$

In this case, we have equation

$$\omega_s(t) = -\frac{1}{s}\sin(st) + \frac{1}{s}\int_0^t q(\tau)\sin\big(s(t-\tau)\big)\omega_s(\tau)d\,\tau. \tag{4.11}$$

Put $\omega_s(t) = \mathrm{e}^{|y|t}F_s(t)$, $(\omega_s)_s'(t,s) = \mathrm{e}^{|y|t}G_s(t)$, $\omega_s'(t) = \mathrm{e}^{|y|t}K_s(t)$. Let $\mu_s = \max_{0\le t\le 1}|F_s(t)|$, $\nu_s = \max_{0\le t\le 1}|G_s(t)|$, $\varkappa_s = \max_{0\le t\le 1}|K_s(t)|$ and $q_0 = 2\int_0^1 |q(\tau)|\,d\tau$. We have

$$\mu_s \le \frac{|s|^{-1}q_0\mu_s}{2} + |s|^{-1}, \qquad \nu_s \le \frac{|s|^{-1}q_0\nu_s}{2} + |s|^{-1}\Big(1 + \frac{q_0\mu_s}{2} + \mu_s\Big),$$
$$\sigma_s \le \frac{|s|^{-1}q_0\sigma_s}{2} + |s|^{-1}\Big(1 + q_0\nu_s + \frac{q_0\mu_s}{2} + 2\nu_s\Big).$$

If $|s| \ge q_0$, then

$$\mu_s \le 2|s|^{-1} = O\big(s^{-1}\big), \qquad \nu_s \le |s|^{-1}(2 + q_0\mu_s + 2\mu_s) = O\big(s^{-1}\big), \tag{4.12}$$
$$\sigma_s \le |s|^{-1}(2 + 2q_0\nu_s + q_0\mu_s + 4\nu_s) = O\big(s^{-1}\big). \tag{4.13}$$

It follows that $\varkappa_s \le 1 + q_0\mu_s/2 = O(1)$ and

$$F_s(t) = -\sin(st)s^{-1}\mathrm{e}^{-|y|t} + O\big(s^{-2}\big), \qquad K_s(t) = -\cos(st)\mathrm{e}^{-|y|t} + O\big(s^{-1}\big),$$
$$G_s(t) = -t\cos(st)s^{-1}\mathrm{e}^{-|y|t} + O\big(s^{-2}\big).$$

So, we prove asymptotic formulas

$$\omega_s(t) = -\sin(st)s^{-1} + O\big(s^{-2}\mathrm{e}^{|y|t}\big), \tag{4.14}$$

$$(\omega_s)'_s(t,s) = -t\cos(st)s^{-1} + O\big(s^{-2}\mathrm{e}^{|y|t}\big), \tag{4.15}$$

$$\omega'_s(t) = -\cos(st) + O\big(s^{-1}\mathrm{e}^{|y|t}\big). \tag{4.16}$$

These formulas hold uniformly for $0 \le t \le 1$.

Let $f \in C^r[a,b]$, $t \in [a,b] \subset [0,1]$. Then, the following asymptotic formulas

$$\int_0^t f(\tau)\cos(2s\tau - st)\,d\tau$$
$$= -\sum_{i=1}^{r-1} \frac{f^{(i-1)}(t) - (-1)^i f^{(i-1)}(0)}{(2s)^i} \cos\left(st + \frac{\pi i}{2}\right) \quad + O\big(s^{-r}\mathrm{e}^{3|y|t}\big), \tag{4.17}$$

$$\int_0^t f(\tau)\sin(2s\tau - st)\,d\tau$$
$$= -\sum_{i=1}^{r-1} \frac{f^{(i-1)}(t) + (-1)^i f^{(i-1)}(0)}{(2s)^i} \sin\left(st + \frac{\pi i}{2}\right) \quad + O\big(s^{-r}\mathrm{e}^{3|y|t}\big) \tag{4.18}$$

are valid.

Under the condition that $q \in C^r[0,1]$, $r \in \mathbb{N}$, the more exact asymptotic formulas can be obtained

$$\omega_s(t) = -\sum_{j=1}^{r+1} p_j(t)\cos\left(st + \frac{1}{2}\pi j\right)s^{-j} + O\big(s^{-(r+2)}\mathrm{e}^{(r+2)|y|t}\big), \tag{4.19}$$

$$\omega'_s(t) = -\sum_{j=0}^{r} \bar{p}_j(t)\cos\left(st + \frac{1}{2}\pi j\right)s^{-j} + O\big(s^{-(r+1)}\mathrm{e}^{(r+2)|y|t}\big), \tag{4.20}$$

$$(\omega_s)'_s(t) = -\sum_{j=1}^{r+1} p^1_j(t)\cos\left(st + \frac{1}{2}\pi(j-1)\right)s^{-j} + O\big(s^{-(r+2)}\mathrm{e}^{(r+2)|y|t}\big), \tag{4.21}$$

where $p_1(t) = -1$, $\bar{p}_0(t) = 1$, $p^1_1(t) = t$.

Now we derive formulas for $p_j,\ j = \overline{2, r+1}$. We can use the mathematical induction. Let us substitute

$$\omega_s(t) = -\sum_{j=1}^{r} p_j(t)\cos\Big(st + \frac{\pi}{2}j\Big)s^{-j} + O\big(s^{-(r+1)}\mathrm{e}^{(r+1)|y|t}\big)$$
$$= -\sum_{j=2}^{r+1} p_{j-1}(t)\sin\Big(st + \frac{\pi}{2}j\Big)s^{-j+1} + O\big(s^{-(r+1)}\mathrm{e}^{(r+1)|y|t}\big) \qquad (4.22)$$

into integral in the right-hand side of (4.11):

$$\sum_{j=2}^{r+1} \frac{-1}{s^j}\int_0^t q(\tau)p_{j-1}(\tau)\sin(st - s\tau)\sin\Big(s\tau + \frac{\pi}{2}j\Big)\,d\tau$$
$$+ O\big(s^{-(r+2)}\mathrm{e}^{(r+2)|y|t}\big).$$

Then we rewrite the sum

$$\sum_{j=2}^{r+1} \frac{1}{2}\int_0^t q(\tau)p_{j-1}(\tau)\,d\tau\,\cos\Big(st + \frac{\pi}{2}j\Big)s^{-j}$$
$$- \sum_{j=2}^{r+1} \frac{\cos(\frac{\pi}{2}j)}{2s^j}\int_0^t q(\tau)p_{j-1}(\tau)\cos(2s\tau - st)\,d\tau$$
$$+ \sum_{j=2}^{r+1} \frac{\sin(\frac{\pi}{2}j)}{2s^j}\int_0^t q(\tau)p_{j-1}(\tau)\sin(2s\tau - st)\,d\tau$$

and apply (4.17)–(4.18) for $p_{j-1} \in C^{r-j+2},\ j = \overline{2, r+1}$:

$$\sum_{j=2}^{r+1} \frac{1}{2}\int_0^t q(\tau)p_{j-1}(\tau)\,d\tau\,\cos\Big(st + \frac{\pi}{2}j\Big)s^{-j} + O\big(s^{-(r+2)}\mathrm{e}^{(r+2)|y|t}\big)$$
$$+ \sum_{j=2}^{r+1}\sum_{i=1}^{r-j+1} \frac{\cos(\frac{\pi}{2}j)}{2s^j}\cdot\frac{(qp_{j-1})^{(i-1)}(t) - (-1)^i(qp_{j-1})^{(i-1)}(0)}{(2s)^i}\cos\Big(st + \frac{\pi}{2}i\Big)$$
$$- \sum_{j=2}^{r+1}\sum_{i=1}^{r-j+1} \frac{\sin(\frac{\pi}{2}j)}{2s^j}\cdot\frac{(qp_{j-1})^{(i-1)}(t) + (-1)^i(qp_{j-1})^{(i-1)}(0)}{(2s)^i}\sin\Big(st + \frac{\pi}{2}i\Big).$$

We look for terms near $s^{-(r+1)}$, i.e. $i+j=r+1$,

$$\begin{aligned}
&\frac{1}{2}\int_0^t q(\tau)p_r(\tau)\,d\tau\,\cos\Bigl(st+\frac{\pi(r+1)}{2}\Bigr)\\
&\quad+\sum_{j=2}^{r}\frac{(qp_{j-1})^{(r-j)}(t)}{2^{r-j+2}}\cos\Bigl(st+\frac{\pi(r+1)}{2}\Bigr)\\
&\quad+\sum_{j=2}^{r}\frac{(-1)^r(qp_{j-1})^{(r-j)}(0)}{2^{r-j+2}}\cos\Bigl(st+\frac{\pi(r+1)}{2}\Bigr)\\
&\quad=-p_{r+1}(t)\cos\Bigl(st+\frac{\pi(r+1)}{2}\Bigr).
\end{aligned}$$

So, we prove recursive formula (4.7) and prove formula (4.8) in the case $l=0$.

The following formulas

$$\begin{aligned}
(\omega_s)_s^{(l)}(t,s)&=-\frac{\partial^l\sin(st)}{\partial s^l}s^{-1}-l(\omega_s)_s^{(l-1)}s^{-1}\\
&\quad+s^{-1}\sum_{j=0}^{l}(-1)^{\lfloor j/2\rfloor}\binom{l}{j}\mathcal{I}_s^j(t,q,(\omega_s)_s^{(l-j)}),\\
(\omega_s')_s^{(l)}(t,s)&=-\frac{\partial^l\cos(st)}{\partial s^l}+\sum_{j=0}^{l}(-1)^{\lfloor j/2+1/2\rfloor}\binom{l}{j}\mathcal{J}_s^j(t,q,(\omega_s)_s^{(l-j)}),\quad l\in\mathbb{N},
\end{aligned}$$

are valid, $\mathcal{I}_s^{2k-2}=I_s^{2k-2}$, $\mathcal{I}_s^{2k-1}=J_s^{2k-1}$, $\mathcal{J}_s^{2k-2}=J_s^{2k-2}$, $\mathcal{J}_s^{2k-1}=I_s^{2k-1}$, $k\in\mathbb{N}$,

$$\begin{aligned}
I_s^k(t,q,f)&=\int_0^t q(\tau)(t-\tau)^k\sin\bigl(s(t-\tau)\bigr)f(\tau)\,d\tau,\\
J_s^k(t,q,f)&=\int_0^t q(\tau)(t-\tau)^k\cos\bigl(s(t-\tau)\bigr)f(\tau)\,d\tau,\\
\tilde{I}_s^k(t,q,f)&=\int_0^t q(\tau)(t-\tau)^k\sin\bigl(s(t-\tau)\bigr)\mathrm{e}^{-|y|(t-\tau)}f(\tau)\,d\tau,\\
\tilde{J}_s^k(t,q,f)&=\int_0^t q(\tau)(t-\tau)^k\cos\bigl(s(t-\tau)\bigr)\mathrm{e}^{-|y|(t-\tau)}f(\tau)\,d\tau.
\end{aligned}$$

Taking derivative with respect to t and s in (4.11), we get

$$\omega_s'(t)=-\cos(st)+J_s^0(t,q,\omega_s),\tag{4.23}$$

$$(\omega_s)_s'(t,s)=-t\cos(st)s^{-1}+\bigl(I_s^0\bigl(t,q,(\omega_s)_s'\bigr)+J_s^1(t,q,\omega_s)-\omega_s(t)\bigr)s^{-1}.\tag{4.24}$$

Let us substitute (4.22) into integral $J_s^0(t,q,\omega_s)$ in the right-hand side of (4.23). Then we get formula (4.9) in the case $l=0$. If we substitute (4.19) and (4.21) into integrals $I_s^0(t,q,(\omega_s)'_s)$ and $J_s^1(t,q,\omega_s)=J_s^0(t,q(\tau)(t-\tau),\omega_s)$, then from (4.24) we get recursive formula ($p_1^1(t)=t$)

$$p_{i+1}^1(t)=-\frac{1}{2}\int_0^t q(\tau)p_i^1(\tau)\,d\tau-\sum_{j=2}^{i}\frac{(qp_{j-1}^1)^{(i-j)}(t)-(-1)^i(qp_{j-1}^1)^{(i-j)}(0)}{2^{i-j+2}}$$
$$+\frac{1}{2}\int_0^t \tilde{q}(\tau)p_i(\tau)\,d\tau-\sum_{j=2}^{i}\frac{(\tilde{q}p_{j-1})^{(i-j)}(t)-(-1)^i(\tilde{q}p_{j-1})^{(i-j)}(0)}{2^{i-j+2}}$$
$$-p_i(t),\quad i=\overline{1,r},\quad \tilde{q}(\tau):=\tilde{q}(t,\tau)=q(\tau)(t-\tau). \tag{4.25}$$

We proved formula (4.8) in the case $l=0,1$ and formula (4.9) in the case $l=0$. The other cases we can prove by mathematical induction by l.

□

Lemma 4.2 (See [4, Lemma 9]) *Let $s\in\mathbb{C}_s$ and $q\in C^r[0,1]$. Then for $|s|\ge q_0$ we have the asymptotic expansions*

$$(\psi_s)_s^{(l)}(t,s)=-\sum_{j=0}^{r}p_j^l(t)\cos\Big(st+\frac{\pi}{2}(j-l)\Big)s^{-j}+O\big(s^{-(r+1)}\mathrm{e}^{(r+2)|y|t}\big), \tag{4.26}$$

$$(\psi'_s)_s^{(l)}(t,s)=-\sum_{j=-1}^{r-1}\bar{p}_j^l(t)\cos\Big(st+\frac{\pi}{2}(j-l)\Big)s^{-j}+O\big(s^{-r}\mathrm{e}^{(r+2)|y|t}\big) \tag{4.27}$$

for $l\in\mathbb{N}_0$, where $p_0^k(t)=-tp_0^{k-1}(t)$, $p_i^k(t)=(1-i)p_{i-1}^{k-1}(t)-tp_i^{k-1}(t)$, $i=\overline{1,r}$, $\bar{p}_{-1}^k(t)=-t\bar{p}_{-1}^{k-1}(t)$, $\bar{p}_i^k(t)=(1-i)\bar{p}_{i-1}^{k-1}(t)-t\bar{p}_i^{k-1}(t)$, $i=\overline{0,r-1}$, $k\in\mathbb{N}$, $\bar{p}_i^0(t)=p_i^{0\prime}(t)-p_{i+1}^0(t)$, $i=\overline{0,r-1}$, $\bar{p}_{-1}^0(t)=1$, and $p_j^0(t)$ is calculated by (4.7) for $i=\overline{0,r-1}$ with $p_0^0(t)=-1$ and $\varrho=-1$.

We will use an additional index to distinguish cases: $p_j^{\mathrm{d},l}(t)$, $\bar{p}_j^{\mathrm{d},l}(t)$ (in Case d), $p_j^{\mathrm{n},l}(t)$, $\bar{p}_j^{\mathrm{n},l}(t)$ (in Case n). So, we get asymptotic expansions for function $\omega_{\alpha s}$.

Lemma 4.3 *Let $s\in\mathbb{C}_s$ and $q\in C^r[0,1]$. Then for $|s|\ge q_0$ we have the asymptotic expansions*

$$(\omega_{\alpha s})_s^{(l)}(t,s)=-\sum_{j=0}^{r}p_j^l(t)\cos\Big(st+\frac{\pi}{2}(j-l)\Big)s^{-j}+O\big(s^{-(r+1)}\mathrm{e}^{(r+2)|y|t}\big), \tag{4.28}$$

$$(\omega'_{\alpha s})^{(l)}_s(t,s) = -\sum_{j=-1}^{r-1} \bar{p}^{\,l}_j(t)\cos\Big(st + \frac{\pi}{2}(j-l)\Big)s^{-j} + \mathcal{O}\big(s^{-r}\mathrm{e}^{(r+2)|y|t}\big) \tag{4.29}$$

for $l \in \mathbb{N}_0$, where

$$p^l_0(t) = \sin\alpha\cdot p^{\mathrm{n},l}_0(t),\ p^l_j(t) = \cos\alpha\cdot p^{\mathrm{d},l}_j(t) + \sin\alpha\cdot p^{\mathrm{n},l}_j(t),\ j=\overline{1,r}, \tag{4.30}$$

$$\bar{p}^l_{-1}(t) = \sin\alpha\cdot \bar{p}^{\mathrm{n},l}_{-1}(t),\ \bar{p}^l_j(t) = \cos\alpha\cdot \bar{p}^{\mathrm{d},l}_j(t) + \sin\alpha\cdot \bar{p}^{\mathrm{n},l}_j(t),\ j=\overline{0,r-1}. \tag{4.31}$$

If $q \in C^1[0,1]$ and $\alpha \in (0,\pi)$, then we have asymptotic expansions:

$$\omega_{\alpha s}(t) = \sin\alpha\cdot\cos(st) + \big(-\cos\alpha + \sin\alpha\, Q(t)\big)\sin(st)s^{-1} + \mathcal{O}\big(s^{-2}\mathrm{e}^{3|y|t}\big),$$
$$\omega'_{\alpha s}(t) = -\sin\alpha\cdot\sin(st)\,s + \big(-\cos\alpha + \sin\alpha\, Q(t)\big)\cos(st) + \mathcal{O}\big(s^{-1}\mathrm{e}^{3|y|t}\big).$$

Lemma 4.4 *Let $x \in \mathbb{R}^+_s$, $\delta \in \mathbb{R}$, $q \in C^r[0,1]$, $Q_j(x)$, $j=\overline{1,r}$ be bounded functions. If $s = x+\delta$, $\delta = \sum_{j=1}^r Q_j(x)x^{-j} + \mathcal{O}\big(x^{-(r+1)}\big)$, then we have the following asymptotic expansion*

$$\omega_{\alpha s}(t) = \sum_{j=0}^{r} R_j(t,x)x^{-j} + \mathcal{O}\big(x^{-(r+1)}\big),$$

$$R_0(t,x) = \sin\alpha\cdot R^{\mathrm{n}}_0(t,x),\quad R_j(t,x) = \cos\alpha\cdot R^{\mathrm{d}}_j(t,x) + \sin\alpha\cdot R^{\mathrm{n}}_j(t,x),\ \ j=\overline{1,r},$$

and functions ($m = \overline{0,r}$)

$$R^{\mathrm{d}}_{m+1}(t,x) = -\sum_{\substack{n_1+\ldots+n_m=l,\ j\ge 1,\\ j+n_1+2n_2+\ldots+mn_m=m+1}} \frac{1}{n_1!\cdots n_m!}p^{\mathrm{d},l}_j(t)\cos(xt + \tfrac{\pi}{2}(j-l))Q_1^{n_1}(x)\cdots Q_m^{n_m}(x),$$

$$R^{\mathrm{n}}_{m}(t,x) = -\sum_{\substack{n_1+\ldots+n_m=l,\ j\ge 0,\\ j+n_1+2n_2+\ldots+mn_m=m}} \frac{1}{n_1!\cdots n_m!}p^{\mathrm{n},l}_j(t)\cos(xt + \tfrac{\pi}{2}(j-l))Q_1^{n_1}(x)\cdots Q_m^{n_m}(x).$$

Proof The proof follows from asymptotic expansions for $\varphi_s(t)$ [3] and $\psi_s(t)$ [4]. □

4.3 Asymptotic Expansions for Characteristic Equations

Substituting $\omega_{\alpha s}(t)$ into (4.3) we get the characteristic equation

$$\begin{aligned} &\text{(Case 1)} && h_\alpha(s) := \omega'_{\alpha s}(1) - \gamma\omega_{\alpha s}(\xi) = 0, \\ &\text{(Case 2)} && h_\alpha(s) := \omega'_{\alpha s}(1) - \gamma\omega'_{\alpha s}(\xi) = 0, \\ &\text{(Case 3)} && h_\alpha(s) := \omega_{\alpha s}(1) - \gamma\omega_{\alpha s}(\xi) = 0. \end{aligned} \tag{4.32}$$

Let's us define functions:

$$\begin{aligned} \text{(Case 1)}\quad & h^l_j(s) := \gamma p^l_j(\xi)\cos\left(\xi s + \tfrac{\pi}{2}(j-l)\right) - \bar{p}^l_j(1)\cos\left(s + \tfrac{\pi}{2}(j-l)\right), \\ & j = \overline{0, r-1}, h^l_{-1}(s) := -\bar{p}^l_{-1}(1)\sin\left(s - \tfrac{\pi}{2}l\right) = (-1)^{l-1} \\ & \sin\alpha\sin\left(s - \tfrac{\pi}{2}l\right), \end{aligned}$$

$$\begin{aligned} \text{(Case 2)}\quad & h^l_j(s) := \gamma\bar{p}^l_j(\xi)\cos\left(\xi s + \tfrac{\pi}{2}(j-l)\right) - \bar{p}^l_j(1)\cos\left(s + \tfrac{\pi}{2}(j-l)\right), \\ & j + 1 = \overline{0, r}, \end{aligned} \tag{4.33}$$

$$\begin{aligned} \text{(Case 3)}\quad & h^l_j(s) := \gamma p^l_j(\xi)\cos\left(\xi s + \tfrac{\pi}{2}(j-l)\right) - p^l_j(1)\cos\left(s + \tfrac{\pi}{2}(j-l)\right), \\ & j = \overline{0, r}, \end{aligned}$$

where functions p^l_j and $\bar{p}^l_j$ are defined by formulas (4.30)–(4.31).

We will use the notation: $\rho = -1$, $a_k := (k - 1/2)\pi$ in Cases 1, 2; $\rho = 0$, $a_k := (k-1)\pi$ in Case 3, $k \in \mathbb{N}$.

Lemma 4.5 *Suppose $|\gamma| < 1$ in Cases 2 and 3. Then $|h^0_\rho(a_k + \imath y)| \geq \varkappa e^{|y|}$, $\varkappa > 0$.*

Lemma 4.6 *Suppose $|\gamma| < 1$ in Cases 2 and 3. There exists $B > 0$ such that $|h^0_\rho(s)| \geq \kappa e^{|y|}$, $\kappa > 0$ for $|y| \geq B$.*

Lemma 4.7 *Let $s \in \mathbb{C}_s$ and $q \in C^r[0,1]$. Then for $|s| \geq q_0$ the asymptotic expansion*

$$h^{(l)}_\alpha(s) = \sum_{j=\rho}^{r+\rho} h^l_j(s)s^{-j} + O\left(s^{-(r+1+\rho)}e^{(r+2)|y|}\right) \tag{4.34}$$

is valid, $l \in \mathbb{N}_0$.

Let us consider positive $s = x > 0$, $q \in C^r[0,1]$, $r \geq 1$. We investigate equation $h_\alpha(x + \delta) = 0$, $\delta \in \mathbb{R}$, with additional condition

$$|h^1_\rho(x)| \geq \varkappa > 0. \tag{4.35}$$

Lemma 4.8 *Suppose $|\gamma| < 1$ in Cases 2 and 3. If $h_\rho^0(x) = 0$, then (4.35) is valid. The constant $\varkappa$ is the same for all such x.*

Let's denote the function $Q_1(x) = -h_{1+\rho}^0(x)\big(h_\rho^1(x)\big)^{-1}$. If functions $Q_1, \dots, Q_{k-1}$ are defined, then we can find functions

$$z_l(x) = \sum_{\substack{n_1+\ldots+n_{k-1}=i,\ j\ge 0,\\ j+n_1+2n_2+\ldots+(k-1)n_{k-1}=l}} -h_{j+\rho}^{i+1}(x)(h_\rho^1(x))^{-1}\frac{Q_1^{n_1}(x)\cdots Q_{k-1}^{n_{k-1}}(x)}{(i+1)n_1!\cdots n_{k-1}!},\ l=\overline{1,k-1},$$

$$Q_k(x) = \sum_{\substack{n_1+\ldots+n_{k-1}=l,\ j>0,\\ j+n_1+2n_2+\ldots+(k-1)n_{k-1}=k}} -h_{j+\rho}^{0}(x)(h_\rho^1(x))^{-1}\frac{l!z_1^{n_1}(x)\cdots z_{k-1}^{n_{k-1}}(x)}{n_1!\cdots n_{k-1}!}.$$

Lemma 4.9 *If $q \in C^r[0,1]$ and $\delta = o(1)$, $h_\rho^0(x) = 0$, then asymptotic expansion*

$$\delta = \sum_{j=1}^{r} Q_j(x)x^{-j} + O\big(x^{-(r+1)}\big) \tag{4.36}$$

is valid, where $Q_j(x)$, $j = \overline{1,r}$, are bounded functions.

Proof Formula (4.36) is valid for $r = 0$. So, $\delta = O\big(x^{-1}\big)$. If $r = 1$, then

$$0 = h(x+\delta) = h(x) + h'(x)\delta + h''(x+\theta\delta)\delta^2/2, \quad \theta \in [0,1],$$

we have $h_\rho^1(x)x^{-\rho}\delta = -h_{1+\rho}^0(x)x^{-1-\rho} + O\big(x^{-2-\rho}\big)$, i.e. $\delta = Q_1(x)x^{-1} + O\big(x^{-2}\big)$.

We derive equations for Q_j, $j = \overline{2,r}$, $r \ge 2$. Suppose that $\delta = \sum_{j=1}^{r-1} Q_j(x)x^{-j} + O\big(x^{-r}\big)$. Substituting (4.34) in the case $y = 0$ into equality

$$0 = h(x+\delta) = h(x) + \delta\sum_{i=0}^{r-1} h^{(i+1)}(x)\frac{\delta^i}{(i+1)!} + \frac{h^{(r+1)}(x+\theta\delta)}{(r+1)!}\delta^{r+1},\ \theta \in [0,1],$$

we get $Z(x)\delta = h_\rho^1(x)x^{-\rho}\Big(\sum_{j=1}^{r} h_j(x)x^{-j} + O\big(x^{-(r+1)}\big)\Big)$, $h_j(x) := -\frac{h_{j+\rho}^0(x)}{(h_\rho^1(x))}$, $1 \le j \le r$, $Z(x) := h_\rho^1(x)x^{-\rho}\big(1 - \sum_{k=1}^{r-1} z_k(x)x^{-k} + O\big(x^{-r}\big)\big)$. So,

$$\delta = \sum_{j=1}^{r}\sum_{l=0}^{r} h_j(x)\sum_{n_1+\ldots+n_{r-1}=l}\frac{l!}{n_1!\cdots n_{r-1}!}\cdot\frac{z_1^{n_1}(x)\cdots z_{r-1}^{n_{r-1}}(x)}{x^{j+n_1+2n_2+\ldots+(r-1)n_{r-1}}} + O\big(x^{-(r+1)}\big).$$

Collecting terms near x^{-r} (i.e. $j + n_1 + 2n_2 + \ldots + (r-1)n_{r-1} = r$) we get $Q_r(x)$. $\square$

4.4 Spectral Asymptotics for Eigenvalues and Eigenfunctions

In this section we assume, that $|\gamma| < 1$ in Cases 2, 3. Let us denote domains $D_k = \{s \in \mathbb{C}\colon\ |x| \le a_k,\ |y| \le a_k\}$, $D_{sk} = \mathbb{C}_s \cap D_k$, $k \in \mathbb{N}$ ($k > 1$ in Case 3), contours $\Gamma_{sk} = \mathbb{C}_s \cap \partial D_k$, and intervals $I_k := \big(a_k,\ a_{k+1}\big) \subset D_{s,k+1} \setminus D_{sk}$, $k \in \mathbb{N}$.

Lemma 4.10 *Suppose $|\gamma| < 1$ in Cases 2 and 3. If $q \in C[0, 1]$, then it follows that the number of zeros of functions $h_\alpha(s)$ and $h^0_\rho(s)s^{-\rho}$ is the same inside Γ_{sk} for sufficiently large k.*

Proof We have $h_\alpha(s) = h^0_\rho(s)s^{-\rho} + O\big(s^{-1-\rho}\mathrm{e}^{|y|}\big)$. Using Lemmas 4.5 and 4.6 we estimate $|O\big(s^{-1-\rho}\mathrm{e}^{|y|}\big)| \le c_1|s|^{-1-\rho}\mathrm{e}^{|y|} < \min\{\kappa, \varkappa\}|s|^{-\rho}\mathrm{e}^{|y|} \le |h^0_\rho(s)s^{-\rho}|$ on the contours Γ_{sk} for sufficiently large k. Therefore, by Rouché theorem it follows that the number of zeros of $h_\alpha(s)$ and $h^0_\rho(s)s^{-\rho}$ are the same inside Γ_{sk} for sufficiently large k. □

From Intermediate Value Theorem at least one root of the function $h_\alpha(s)$ lies in I_k for sufficiently large k. So, s_k is real root for such k.

We have $s_k \sim x_k \sim \pi k$ (as $k \to \infty$). Then $h_\alpha(s_k)\cdot s_k^\rho = h^0_\rho(s_k) + O\big(k^{-1}\big) = 0$ and $\lim_{k\to\infty} h^0_\rho(s_k) = 0$. The function h^0_ρ is analytic and has one root in I_k. Additionally, $|h^1_\rho(x_k)| \ge \kappa > 0$ (see Lemma 4.8). Therefore, $s_k \to x_k$ as $k \to \infty$ or $\delta_k = o(1)$.

Theorem 4.1 *Let $q \in C^r[0, 1]$. For eigenvalues $\lambda_k = s_k^2$ and eigenfunctions u_k of problem (4.1)–(4.3), we have the asymptotic expansions*

$$s_k = x_k + \sum_{j=1}^{r} Q_j(x_k)x_k^{-j} + O\big(k^{-(r+1)}\big), \tag{4.37}$$

$$u_k(t) = \sum_{j=0}^{r} R_j(t, x_k)x_k^{-j} + O\big(k^{-(r+1)}\big) \tag{4.38}$$

for sufficiently large k.

References

1. Levitan, B.M., Sargsjan, I.S.: Sturm-Liouville and Dirac Operators. Kluwer, Dordrecht (1991)
2. Şen, E., Štikonas, A.: Asymptotic distribution of eigenvalues and eigenfunctions of a nonlocal boundary value problem. Math. Model. Anal. **26**(2), 253–266 (2021). https://doi.org/10.3846/mma.2021.13056
3. Štikonas, A., Şen, E.: Asymptotic analysis of Sturm–Liouville problem with nonlocal integral-type boundary condition. Nonlinear Anal. Model. Control. **26**(5), 969–991 (2021). https://doi.org/10.15388/namc.2021.26.24299
4. Štikonas, A., Şen, E.: Asymptotic analysis of Sturm–Liouville problem with Neumann and nonlocal two-point boundary conditions. Lith. Math. J. **62**(4), 519–541 (2022). https://doi.org/10.1007/s10986-022-09577-6

5. Štikonas, A., Şen, E.: Asymptotic analysis of Sturm–Liouville problem with Dirichlet and nonlocal two-point boundary conditions. Math. Model. Anal. **28**(2), 308–329 (2023). https://doi.org/10.3846/mma.2023.17617
6. Titchmarsh, E.C.: Eigenfunction Expansions Associated with Second-Order Differential Equations. Clarendon Press, Oxford (1946)

Chapter 5
Characterization of the Constant Sign of a Class of Periodic and Neumann Green's Functions via Spectral Theory

Alberto Cabada and Lucía López-Somoza

Abstract In this paper, we characterize the regions of constant sign of the Green's fucntions related to operator $T_n[p, M]\, u(t) = u^{(n)}(t) + p\, u^{(n-2)}(t) + M\, u(t)$, with n being an even number, $n \geq 4$, and $p \leq 0$, coupled to periodic or Neumann boundary conditions. The results generalize the situation considered in Cabada and López-Somoza (Differ Equ Appl 14(2):335–347, 2022) for the particular case of $p = 0$.

5.1 Introduction and Preliminaries

The study of nonlinear Boundary Value Problems is closely related to the constant sign of the solutions related to the linear part of the studied equation. Such constant sign is fundamental to develop the method of lower and upper solutions [4], the monotone iterative techniques [5] or the existence of solutions defined in suitable cones [7]. Such property is equivalent to the constant sign of the related Green function [8]. Due to the difficulty in obtaining the exact expression of such functions and that, in case of having such expression, it is very difficult to manage it, to develop a theory that allows us to know when the Green's function has constant sign in a direct way, without necessity of obtaining its expression, is of a great importance (see [6, 8, 9] and references therein for the Hill's equation). In this paper, we consider an even order periodic equation and extend the results obtained by the authors in [3] to a more general situation. So, let $T > 0$, $p \leq 0$, $M \in \mathbb{R}$ and $n \in \mathbb{N}$ an even number, $n \geq 4$, be given. Consider the n-th order linear operator

$$T_n[p, M]\, u(t) := u^{(n)}(t) + p\, u^{(n-2)}(t) + M\, u(t), \qquad \text{for all } t \in I := [0, T]. \tag{5.1}$$

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Definition 5.1 Given a Banach space $X \subset C^n(I)$, operator $T_n[p, M]$ is said to be nonresonant in X if and only if the homogeneous equation

$$T_n[p, M]\, u(t) = 0 \quad \text{for all } t \in I, \qquad u \in X$$

has only the trivial solution.

Definition 5.2 Given a Banach space $X \subset C^n(I)$ and $\bar{M} \in \mathbb{R}$, we say that $\bar{\lambda} \in \mathbb{R}$ is an eigenvalue of operator $T_n[p, \bar{M}]$ in X if and only if the homogeneous equation

$$T_n[p, \bar{M} + \bar{\lambda}]\, u(t) = 0 \quad \text{for all } t \in I, \qquad u \in X$$

has non trivial solutions.

It is very well known that if operator $T_n[p, M]$ is nonresonant in X then, for any $\sigma \in C(I)$, the non homogeneous problem

$$T_n[p, M]\, u(t) = \sigma(t) \quad \text{for all } t \in I, \quad u \in X$$

has a unique solution given by

$$u(t) = \int_0^T G[p, M, T](t, s)\, \sigma(s)\, ds, \qquad \text{for all } t \in I.$$

Function $G[p, M, T]$ is the so-called Green's function related to operator $T_n[p, M]$ on X. See [1] for details.

Now we introduce the concept of inverse positive and inverse negative operators.

Definition 5.3 Operator $T_n[p, M]$ is inverse positive (negative) in X if and only if the related Green's function $G[p, M, T] \geq 0$ ($G[p, M, T] \leq 0$) on $I \times I$.

Thus, the Banach space X for the periodic boundary conditions becomes

$$X^n_{P,T} = \left\{ u \in C^n(I) : \; u^{(j)}(0) = u^{(j)}(T), \; j = 0, \ldots, n-1 \right\},$$

and the corresponding Green's function is denoted by $G_P[p, M, T]$.

As a direct consequence of [1, Theorems 1.8.5 and 1.8.9, Lemmas 1.8.25 and 1.8.33], since $M = 0$ is the main eigenvalue of $T_n[p, 0]$ in $X^n_{P,T}$ (the corresponding eigenfunctions have constant sign in I), which implies that $G_P[p, 0, T]$ does not exist, we deduce that for any $p \leq 0$ given, there are $M_1(p) \leq 0 \leq M_2(p)$ (or $M_1(p) = -\infty$ and/or $M_2(p) = \infty$) for which $M\, G_P[p, M, T] > 0$ in $I \times I$ if and only if $M \in (M_1(p), M_2(p)) \backslash \{0\}$. Moreover, if $M_1(p) < 0$ ($M_2(p) > 0$), then it is not an eigenvalue of operator $T_n[p, 0]$ in $X^n_{P,T}$ and the nonpositive function $G_P[p, M_1(p), T]$ (nonnegative function $G_P[p, M_2(p), T]$) vanishes at some point of $I \times I$. Furthermore, function $G_P[p, M, T]$ is monotone decreasing with respect to M on $[M_1(p), 0)$ ($(0, M_2(p)]$).

In the sequel we will prove that $M_1(p) < 0 < M_2(p)$, i.e., the set of values for which operator $T_n[p, M]$ is inverse positive (negative) in $X^n_{P,T}$ is not empty. Before this, we will enunciate the following very well know result (see [1, Section 1.9] and references therein):

Lemma 5.1 *The following properties are verified:*

1. *Operator $T_2[0, M]$ is inverse negative in $X^2_{P,T}$ if and only if $M < 0$.*
2. *Operator $N_{A,B}\, u = u'' - 2\, A\, u' + (A^2 + B^2)\, u$ is inverse positive in $X^2_{P,T}$ if and only if $0 < B \le \dfrac{\pi}{T}$.*
3. *If the linear operators L_1 and L_2 are either both inverse positive or both inverse negative operators in $X^n_{P,T}$ and $X^m_{P,T}$ respectively, then $L_1 \circ L_2$ is inverse positive in $X^{n+m}_{P,T}$.*
4. *If the linear operators L_1 and L_2 are, respectively, inverse positive in $X^n_{P,T}$ and inverse negative in $X^m_{P,T}$, then $L_1 \circ L_2$ is inverse negative in $X^{n+m}_{P,T}$.*

Let us prove now the aforementioned result. The proof uses similar arguments to the ones used in (see [1, Section 1.9] and references therein) for the particular case of $p = 0$. It consist of rewriting the n-order operator as the composition of $n/2$ operators of second order.

Lemma 5.2 *Let $n \ge 4$ be an even number and $p \le 0$. Then there exist $M_1(p) < 0 < M_2(p)$ such that $T_n[p, M]$ is inverse negative in $X^n_{P,T}$ if $[M_1(p), 0)$ and inverse positive in $X^n_{P,T}$ if $(0, M_2(p)]$.*

Proof Let us consider the polynomial function $f_M(\lambda) := \lambda^n + p\, \lambda^{n-2} + M$. Since n is even, it is obvious that $f_M(\lambda) = 0$ if and only if $f_M(-\lambda) = 0$, i.e., its roots are symmetric. Let us study such roots for values of M near to 0:

- If $M = 0$, then $f_0(\lambda) = \lambda^{n-2}\, (\lambda^2 + p)$ has a root of multiplicity $n - 2$ at $\lambda = 0$ and two simple roots $\lambda = \sqrt{-p}$ and $\lambda = -\sqrt{-p}$.
- If $M < 0$, then $f_M(\lambda)$ has two simple real roots $\lambda_1 > 0 > -\lambda_1$, and $n - 2$ conjugated complex roots: $\alpha_j \pm i\, \beta_j$, $j = 1, \dots, (n-2)/2$.
- If $M > 0$ is next to 0, then using the continuity of the roots of the polynomial, $f_M(\lambda)$ has four simple real roots $\lambda_1 > 0 > -\lambda_1$, $\lambda_2 > 0 > -\lambda_2$, and $n - 4$ conjugated complex roots: $\tilde{\alpha}_j \pm i\, \tilde{\beta}_j$, $j = 1, \dots, (n-4)/2$.

As a consequence, we have that if $M < 0$, then

$$f_M(\lambda) = (\lambda^2 - \lambda_1^2) \prod_{j=1}^{\frac{n-2}{2}} ((\lambda - \alpha_j)^2 + \beta_j^2),$$

and so

$$T_n[p, M] \equiv \overline{T}_1 \circ T_1 \circ \cdots \circ T_{\frac{n-2}{2}},$$

where $\overline{T}_1 u = u'' - \lambda_1^2\, u$ and $T_j u = u'' - 2\,\alpha_j\, u' + (\alpha_j^2 + \beta_j^2)\, u$.

From Lemma 5.1 we have that $\overline{T}_1$ is inverse negative in $X^2_{P,T}$. Moreover, if M is near to 0, by the continuity of the roots of the polynomial, we have that $0 < \beta_j \le \frac{\pi}{T}$ and so T_j is inverse positive in $X^2_{P,T}$ for all $j = 1, \dots, \frac{n-2}{2}$. Thus, using Lemma 5.1 again, we conclude that $T_n[p, M]$ is inverse negative in $X^n_{P,T}$ for $M < 0$ next to 0.

On the other hand, if $M > 0$ is next to 0, then

$$f_M(\lambda) = (\lambda^2 - \lambda_1^2)\,(\lambda^2 - \lambda_2^2)\,\prod_{j=1}^{\frac{n-4}{2}} ((\lambda - \tilde{\alpha}_j)^2 + \tilde{\beta}_j^2),$$

and so

$$T_n[p, M] \equiv \overline{T}_1 \circ \overline{T}_2 \circ \tilde{T}_1 \circ \cdots \circ \tilde{T}_{\frac{n-4}{2}},$$

where $\overline{T}_1 u = u'' - \lambda_1^2\, u$, $\overline{T}_2 u = u'' - \lambda_2^2\, u$ and $\tilde{T}_j u = u'' - 2\,\tilde{\alpha}_j\, u' + (\tilde{\alpha}_j^2 + \tilde{\beta}_j^2)\, u$.

From Lemma 5.1, both $\overline{T}_1$ and $\overline{T}_2$ are inverse negative in $X^2_{P,T}$. Moreover, if M is near to 0 it holds that $0 < \tilde{\beta}_j \le \frac{\pi}{T}$ and so $\tilde{T}_j$ is inverse positive in $X^2_{P,T}$ for all $j = 1, \dots, \frac{n-4}{2}$ and, as a consequence, Lemma 5.1 implies that $T_n[p, M]$ is inverse positive in $X^n_{P,T}$ for $M > 0$ next to 0. □

Since all the coefficients in operator $T_n[p, M]$ are constant, we are in conditions to apply the following result (see [1, Section 1.4] and references therein), that ensures that $G_P[p, M, T]$ is constant over the straight lines of slope equals to 1.

Lemma 5.3 *The Green's function $G_P[p, M, T]$ related to the operator $T_n[p, M]$ in $X^n_{P,T}$ is given by the following expression:*

$$G_P[p, M, T](t, s) = \begin{cases} G_P[p, M, T](t - s, 0), & 0 \le s \le t \le T, \\ G_P[p, M, T](T + t - s, 0), & 0 \le t \le s \le T. \end{cases}$$

Moreover, $r_M(t) := G_P[p, M, T](t, 0)$ is the unique solution of the following problem:

$$\begin{cases} T_n[p, M]\, r_M(t) = 0, \ t \in I, \\ r_M^{(i)}(0) - r_M^{(i)}(T) = 0, \ i = 0, \dots, n-2, \\ r_M^{(n-1)}(0) - r_M^{(n-1)}(T) = 1. \end{cases} \tag{5.2}$$

As it is stated on the proof of [1, Corollary 1.4.12] for a more general situation, it is immediate to verify that if $n = 2k$ is even, then $r_M(t) = r_M(T - t)$ for all $t \in I$. Notice that, as a direct consequence, we deduce that

$$r_M^{(j)}(t) = (-1)^j\, r_M^{(j)}(T - t) \qquad \text{for all } t \in I \text{ and } j \in \{0, 1, \dots, 2k\}. \tag{5.3}$$

In particular,

$$r_M^{(2j+1)}(T/2) = 0 \qquad \text{for all} \;\; j \in \{0, 1, \ldots, k-1\}, \tag{5.4}$$

$$r_M^{(2j+1)}(0) = r_M^{(2j+1)}(T) = 0, \qquad j \in \{0, 1, \ldots, k-2\}, \tag{5.5}$$

$$r_M^{(2k-1)}(0) = 1/2 \qquad \text{and} \qquad r_M^{(2k-1)}(T) = -1/2. \tag{5.6}$$

Now, for any even natural number $n = 2k$, we will denote by $G_N[p, M, T]$ the Green's function related to the operator $T_n[p, M]$ coupled to the so-called Neumann boundary conditions:

$$X_{N,T}^n = \left\{ u \in C^n(I) : \; u^{(2j+1)}(0) = u^{(2j+1)}(T) = 0, \; j = 0, \ldots, k-1 \right\}.$$

As it is shown in [2, Theorem 3], in case of constant coefficients, the regions of constant sign of the Green's functions related to periodic and Neumann conditions coincide in intervals of double length. The result is the following:

Theorem 5.1 *The following property is fulfilled:*
$G_P[p, M, 2T] \leq 0$ *(*$G_P[p, M, 2T] \geq 0$*) on* $[0, 2T] \times [0, 2T]$ *if and only if* $G_N[p, M, T] \leq 0$ *(*$G_N[p, M, T] \leq 0$*) on* $I \times I$.

5.2 Characterization of Constant Sign of Periodic and Neumann Green's Functions

In this section, we will extend Theorems 2 and 4 in [3] to the case $p < 0$. In particular, we will consider problems of order $n = 2k \geq 4$. The case $n = 2$ is considered in [3] but such situation makes no sense in the context of this paper because in such a case $n - 2 = 0$ and so the problem would be reduced to the one studied in [3]. The obtained result is the following.

Theorem 5.2 *Let* $n = 2k$ *with* $k \in \mathbb{N}$, $k \geq 2$, *and* $p \leq 0$. *Then the following properties hold:*

1.- The Green's function related to operator $T_n[p, M]$ *on* $X_{P,T}^n$ *is nonnegative on* $I \times I$ *(and strictly positive on* $I \times I$ *if* M *is on the interior of the intervals) if and only if the following conditions are fulfilled:*

1. $k = 2l + 1$ *for some* $l \geq 1$, *and* $M \in (0, M_2(p)]$, *being* $M_2(p)$ *the least positive eigenvalue of problem*

$$\begin{cases} r^{(n)}(t) + p\, r^{(n-2)}(t) = 0, \; t \in [0, T/2], \\ r(0) = 0, \\ r^{(2j+1)}(0) = 0, \; j \in \{0, 1, \ldots, k-2\}, \\ r^{(2j+1)}(T/2) = 0, \; j \in \{0, 1, \ldots, k-1\}. \end{cases} \tag{5.7}$$

2. *$k = 2l$ for some $l \geq 1$, and $M \in (0, M_2(p)]$, being $M_2(p)$ the least positive eigenvalue of problem*

$$\begin{cases} r^{(n)}(t) + p\, r^{(n-2)}(t) = 0, \ t \in [0, T/2], \\ r(T/2) = 0, \\ r^{(2j+1)}(0) = 0, \ j \in \{0, 1, \dots, k-2\}, \\ r^{(2j+1)}(T/2) = 0 = 0, \ j \in \{0, 1, \dots, k-1\}. \end{cases} \tag{5.8}$$

2.- *The Green's function related to operator $T_n[p, M]$ on $X^n_{P,T}$ is nonpositive on $I \times I$ (and strictly negative on $I \times I$ if M is on the interior of the intervals) if and only if the following conditions are fulfilled:*

1. *$k = 2l + 1$ for some $l \geq 1$, and $M \in [M_1(p), 0)$, being $M_1(p)$ the biggest negative eigenvalue of Problem (5.8).*
2. *$k = 2l$ for some $l \geq 1$, and $M \in [M_1(p), 0)$, being $M_1(p)$ the biggest negative eigenvalue of Problem (5.7).*

Proof Let $p \leq 0$ be fixed. From Lemma 5.3, it is clear that the sign of the Green's function on $I \times I$ is characterized by the sign of the function r_M on I. Moreover, from Lemma 5.2 we know that there exist $M_1(p) < 0 < M_2(p)$ for which $M\, r_M > 0$ in I if and only if $M \in (M_1(p), M_2(p)) \backslash \{0\}$. In particular, since r_M is monotone decreasing with respect to $M \in [M_1(p), 0) \cup (0, M_2(p)]$, we have that our problem is reduced to find the exact values of $M_1(p) < 0 < M_2(p)$, that satisfy that they are the unique real constants for which r_M has constant sign on I and vanishes at some point in I.

It is important to point out that if $M \in [M_1(p), M_2(p)] \backslash \{0\}$ then M is not an eigenvalue of operator $u^{(n)} + p\, u^{(n-2)}$ on $X^n_{P,T}$. As a direct consequence, identities (5.4) and (5.5) imply that r_M satisfies the two last sets of boundary conditions imposed in Problems (5.7) and (5.8) for all M in such intervals.

Let us define $v(t) := r''_M(t) + p\, r_M(t)$, $t \in I$. Taking into account (5.3), it is immediate to verify that v satisfies the same property, that is:

$$v^{(j)}(t) = (-1)^j\, v^{(j)}(T - t) \qquad \text{for all } t \in I \text{ and } j \in \{0, 1, \dots, 2k\}. \tag{5.9}$$

In particular,

$$v^{(2j+1)}(T/2) = 0 \qquad \text{for all } \ j \in \{0, 1, \dots, k-1\}. \tag{5.10}$$

Moreover, from (5.5) and (5.6), we deduce that

$$v^{(2j+1)}(0) = v^{(2j+1)}(T) = 0, \qquad j \in \{0, 1, \dots, k-3\} \cup \{k-1\} \tag{5.11}$$

and

$$v^{(2k-3)}(0) = 1/2 \qquad \text{and} \qquad v^{(2k-3)}(T) = -1/2. \tag{5.12}$$

Finally, by integration, we reach to

$$\int_0^1 v(s)\,ds = \int_0^1 r''_M(s)\,ds + p\int_0^1 r_M(s)\,ds = p\int_0^1 r_M(s)\,ds. \tag{5.13}$$

Now, if $M \in (M_1(p), M_2(p))\backslash\{0\}$, since $M\,r_M(t) > 0$ for all $t \in I$, we have that $v^{(2\,k-2)}(t) = r_M^{(2\,k)}(t) + p\,r_M^{(2\,k-2)}(t) = -M\,r_M(t) < 0$ for all $t \in I$. Thus, we deduce that $v^{(2\,k-3)}$ is strictly decreasing on I. Moreover, as a direct consequence of (5.10) and (5.12), we conclude that

$$v^{(2\,k-3)}(t) > 0 \quad \text{for all } t \in (0, T/2) \text{ and } v^{(2\,k-3)}(t) < 0 \quad \text{for all } t \in (T/2, T).$$

Therefore, $v^{(2\,k-4)}$ is strictly increasing on $(0, T/2)$ and strictly decreasing in $(T/2, T)$.

Now, if $\mathbf{k} = \mathbf{2}$ (i.e. $2k - 4 = 0$), we shall distinguish two different cases:

Case 1: $\mathbf{M} \in (\mathbf{0}, \mathbf{M_2(p)}]$
In this case, it is clear that r_M is nonnegative on I and so, from (5.13), we know that either v is nonpositive or changes its sign on I.

Now, let $t_0 \in I$ be such that $r_{M_2(p)}(t_0) = 0$ (by symmetry, we may assume that $t_0 \in [0, T/2]$). Since $r_{M_2(p)}$ is nonnegative on I, t_0 is a minimum of $r_{M_2(p)}$ and so $r'_{M_2(p)}(t_0) = 0$ and $r''_{M_2(p)}(t_0) \geq 0$. In such a case, $v(t_0) = p\,r''_{M_2(p)}(t_0) \geq 0$. Therefore, if $r_{M_2(p)}$ vanishes at some point t_0, v must change its sign on I, that is, there exists $t_1 \in (0, T/2)$ such that

$$v(t) > 0 \text{ for all } t \in (t_1, T - t_1) \text{ and } v(t) < 0 \text{ for all } t \in [0, t_1) \cup (T - t_1, T]$$

with $t_0 \in [t_1, T - t_1)$.

Moreover, for $t \in (t_1, T - t_1)$, it occurs that

$$r''_{M_2(p)}(t) \geq r''_{M_2(p)}(t) + p\,r_{M_2(p)}(t) = v(t) > 0$$

and so $r'_{M_2(p)}$ is strictly increasing in $(t_1, T - t_1)$.
Since $r'_{M_2(p)}(T/2) = r'_{M_2(p)}(t_0) = 0$, it must occur that $t_0 = T/2$. Thus, $M_2(p)$ is the least positive eigenvalue of problem

$$r^{(4)}(t) + p\,r''(t) = 0,\ t \in [0, T/2], \quad r(T/2) = r'(0) = r'(T/2) = r'''(T/2) = 0.$$

Case 2: $\mathbf{M} \in [\mathbf{M_1(p)}, \mathbf{0})$
Similarly, in this case it occurs that r_M is nonpositive on I and so (5.13) implies that either v is nonnegative or changes sign on I.

Taking again $t_0 \in I$ such that $r_{M_1(p)}(t_0) = 0$ and reasoning as in previous case, it can be proved that $v(t_0) \leq 0$ and, consequently, v must change its sign on I. Thus, there exists $t_1 \in (0, T/2)$ such that

$$v(t) > 0 \text{ for all } t \in (t_1, T - t_1) \text{ and } v(t) < 0 \text{ for all } t \in [0, t_1) \cup (T - t_1, T]$$

with $t_0 \in [0, t_1]$.

Now, for $t \in [0, t_1)$,

$$r''_{M_1(p)}(t) \leq r''_{M_1(p)}(t) + p\, r_{M_1(p)}(t) = v(t) < 0$$

and $r'_{M_1(p)}$ is strictly decreasing in $[0, t_1)$. Since $r'_{M_1(p)}(0) = r'_{M_1(p)}(t_0) = 0$, it must occur that $t_0 = 0$. Thus, $M_1(p)$ is the biggest negative eigenvalue of problem

$$r^{(4)}(t) + p\, r''(t) = 0,\ t \in [0, T/2], \quad r(0) = r'(0) = r'(T/2) = r'''(T/2) = 0.$$

On the other hand, if $\mathbf{k} > \mathbf{2}$, (5.11) together to the monotonicity properties previously proved for $v^{(2k-4)}$, implies that $v^{(2k-4)}$ must change its sign on I and so, we deduce that there exists $t_1 \in (0, T/2)$ such that

$$v^{(2k-4)}(t) > 0 \text{ for all } t \in (t_1, T - t_1) \text{ and } v^{(2k-4)}(t) < 0 \text{ for all } t \in [0, t_1) \cup (T - t_1, T].$$

Thus, together with (5.9), (5.10) and (5.11) imply that

$$v^{(2k-5)}(t) < 0 \text{ for all } t \in (0, T/2) \text{ and } v^{(2k-5)}(t) > 0 \text{ for all } t \in (T/2, T),$$

and, as a consequence, $v^{(2k-6)}$ is strictly decreasing on $(0, T/2)$ and strictly increasing in $(T/2, T)$.

Now, if $\mathbf{k} = \mathbf{3}$, we shall distinguish two cases and reasoning similarly to the case $k = 2$, we arrive at the following results:

Case 1: $\mathbf{M} \in (\mathbf{0}, \mathbf{M_2(p)}]$
Reasoning as in Case 1 for $k = 2$, we deduce that if t_0 is a minimum of $r_{M_2(p)}$ ($r_{M_2(p)}(t_0) = 0$) then v must change its sign on I. In particular, there exists $t_1 \in (0, T/2)$ such that

$$v(t) < 0 \text{ for all } t \in (t_1, T - t_1) \text{ and } v(t) > 0 \text{ for all } t \in [0, t_1) \cup (T - t_1, T]$$

with $t_0 \in [0, t_1)$. Also, it occurs that $r''_{M_2(p)}(t) > 0$ for $t \in [0, t_1)$ and so $r'_{M_2(p)}$ is strictly increasing in such interval and, since $r'_{M_2(p)}(0) = r'_{M_2(p)}(t_0) = 0$, necessarily $t_0 = 0$. This way, we conclude that $M_2(p)$ is the least positive eigenvalue of problem

$$\begin{cases} r^{(6)}(t) + p\, r^{(4)}(t) = 0,\ t \in [0, T/2], \\ r(0) = r'(0) = r'''(0) = r'(T/2) = r'''(T/2) = r^{(5)}(T/2) = 0. \end{cases}$$

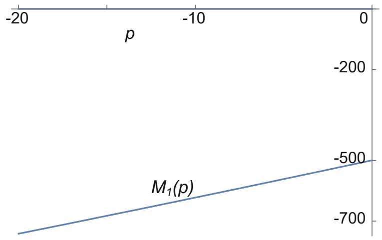

Fig. 5.1 Values of $M_1(p)$ for $T = 1$

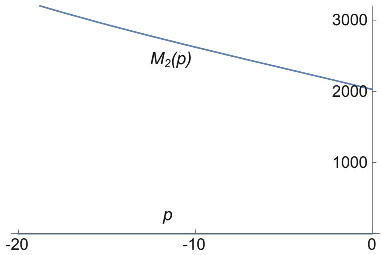

Fig. 5.2 Values of $M_2(p)$ for $T = 1$

Case 2: $\mathbf{M} \in [\mathbf{M_1(p)}, \mathbf{0})$
Using similar arguments to previous cases, we deduce that $M_1(p)$ is the biggest negative eigenvalue of problem

$$\begin{cases} r^{(6)}(t) + p\, r^{(4)}(t) = 0, \ t \in [0, T/2], \\ r'(0) = r'''(0) = r(T/2) = r'(T/2) = r'''(T/2) = r^{(5)}(T/2) = 0. \end{cases}$$

For $\mathbf{k} > \mathbf{3}$, by (5.11), $v^{(2k-6)}$ must change its sign on I and, reasoning as before, we may deduce that

$$v^{(2k-7)}(t) > 0 \text{ for all } t \in (0, T/2) \text{ and } \ v^{(2k-7)}(t) < 0 \text{ for all } t \in (T/2, T).$$

In this case, we are in the same situation as $v^{(2k-3)}$ and so the result holds by recurrence. □

To obtain a numerical approach of the eigenvalues is very simple. It consists on looking for the zeros of the corresponding Wronskians. In [3, Section 5] it is explained in detail how to do it. In Figs. 5.1 and 5.2, the values of $M_1(p)$ and $M_2(p)$ are plotted for $p \in [-20, 0]$ and $T = 1$.

Remark 5.1 It is immediate to verify that v is a solution of $T_n[p, M]\, v = 0$ on $[a, b]$, together the boundary conditions (5.7) or (5.8) by replacing 0 and T by a and b respectively, if and only if $u(t) := v((b-a)t/T + a))$, $t \in I$, is a solution of $T_n[p\,(T/(b-a))^2, M(T/(b-a))^n]\, u = 0$ on I. So, as a direct consequence, we obtain that the eigenvalues of such problems $\lambda_j(p, a, b)$ (with obvious notation) satisfy the equality

$$\lambda_j(p, a, b) = \lambda_j(p\,(T/(b-a))^2, 0, T)\,(T/(b-a))^n, \quad j \in \mathbb{N}.$$

As it has been done in [3], as a direct consequence of Theorem 5.1 we can rewrite the results for the periodic problem obtained in Theorem 5.2 to the Neumann boundary value problem. In this case, the result is the following one.

Theorem 5.3 *Let $n = 2k$ with $k \in \mathbb{N}$, $k \geq 2$, and $p \leq 0$. Then the following properties are fulfilled:*

1.- The Green's function related to operator $T_n[p, M]$ on the space $X^n_{N,T}$, of functions satisfying T–Neumann boundary conditions, is nonnegative on $I \times I$ (and strictly positive on $I \times I$ if M is on the interior of the intervals) if and only if $M \in (0, M_2^N(p, 0, T)]$, being $M_2^N(p, 0, T)$ the least positive eigenvalue of either Problem (5.7) or (5.8), as appropriate, by replacing $T/2$ instead of T.

2.- The Green's function related to operator $T_n[p, M]$ on the space $X^n_{N,T}$, of functions satisfying T–Neumann boundary conditions, is nonpositive on $I \times I$ (and strictly negative on $I \times I$ if M is on the interior of the intervals) if and only if $M \in [M_1^N(p, 0, T), 0)$, being $M_1^N(p, 0, T)$ the biggest negative eigenvalue of either Problem (5.7) or (5.8), as appropriate, by replacing $T/2$ instead of T .

Arguing as in Remark 5.1, we deduce that (with obvious notation)

$$M_i^N(p, 0, T) = M_i(p/4, 0, T)/2^n, \quad i = 1, 2.$$

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References

1. Cabada, A.: Green's Functions in the Theory of Ordinary Differential Equations. Springer Briefs in Mathematics. Springer, New York (2014)
2. Cabada, A., López-Somoza, L.: Relationship between Green's functions for even order linear boundary value problems. In: Nonlinear Analysis and Boundary Value Problems, pp. 243–263 Springer Proceedings in Mathematics & Statistics, vol. 292. Springer, Cham (2019)
3. Cabada, A., López-Somoza, L.: Spectral characterization of the constant sign Green's functions for periodic and Neumann boundary value problems of even order. Differ. Equ. Appl. **14**(2), 335–347 (2022)
4. De Coster, C., Habets, P.: Two-Point Boundary Value Problems: Lower and Upper Solutions. Mathematics in Science and Engineering, vol. 205. Elsevier, Amsterdam (2006)
5. Heikkilä, S., Lakshmikantham, V.: Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, New York (1994)
6. Torres, P. J.: Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem. J. Differ. Equ. **190**(2), 643–662 (2003)
7. Webb, J.R.L., Infante, G.: Positive solutions of nonlocal boundary value problems: a unified approach. J. Lond. Math. Soc. **74**(3), 673–693 (2006)
8. Zhang, M.: Optimal conditions for maximum and anti-maximum principles of the periodic solution problem. Bound. Value Probl. **2010**, 410986, 26pp. (2010)
9. Zhang, M., Li, W.: A Lyapunov-type stability criterion using L^α norms. Proc. Am. Math. Soc. **130**(11), 3325–3333 (2002)

Chapter 6
Results for Multidimensional Hardy Operator Using Domain Partitions

Elena Lomakina and Kairat Mynbaev

Abstract We consider a Hardy-type integral operator T associated with a family of open subsets $\Omega(t)$ of an open set Ω in a Hausdorff topological space X. We find necessary and sufficient conditions for the boundedness and compactness of the operator T and two-sided estimates for its approximation numbers. All results are obtained using domain partitions, thus providing a road map for generalizing many one-dimensional results to a Hausdorff topological space.

6.1 Introduction

A one-dimensional Hardy inequality,

$$\left[\int_0^\infty u(x)\left(\int_0^x f(y)\,dy\right)^q d\mu(x)\right]^{1/q} \le C\left(\int_0^\infty f^p(y)\,v(y)\,d\nu(y)\right)^{1/p}$$

has been studied in detail and complete characterizations of its validity for all non-negative functions f have been obtained in terms of pairs of weights u, v and measures μ, ν for all pairs of exponents $p, q \in (1, \infty)$, see [9–12, 14] for the history and extensive references.

In the one-dimensional case, most researchers have utilized tools of one-dimensional calculus, such as integration by parts [24]. The lack of such tools has been the main obstacle on the way to multidimensional results. Some general results for $p \le q$ and Banach function spaces have been established in [7]. Obtaining full characterizations has been facilitated by the possibility of reducing

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the multidimensional case to the one-dimensional [3, 17, 18, 22, 26, 27]. The result by Sawyer [20] does not allow reduction to one dimension but is limited to a quadrant on the plane R^2.

In a recent paper, Sinnamon [23] suggested a very general method that covers totally ordered sets of domains on a measure space. The method relies on a non-increasing rearrangement involving the weights and measures and reduces the multidimensional case to the one-dimensional. The analysis of ordered cores is of independent interest.

It is desirable to have everything expressed in terms of original weights and measures for some applications, the most important examples being the Hardy-Steklov type operator [8] and the Hardy inequality on cones of monotone functions [21, 25]. In Sinnamon's method, one additional step is required to derive the criteria in terms of original weights and measures from his one-dimensional formulations. Mynbaev [15] has obtained results in terms of original weights and measures under the assumptions on the domains that are close to the ones imposed by Sinnamon (see [15, Remark 1] for a more detailed comparison with Sinnamon's paper).

Here, we develop a different approach to the norm estimation, compactness conditions, and bounds for approximation numbers using domain partitions, a technique widely applied in one dimension. The boundedness criteria obtained below can be derived from both [23] and [15]. Nevertheless, we give independent proofs of boundedness, compactness, and estimates of approximation numbers to show that domain partitions, combined with the conditions on the operator T imposed here, allow one to extend many of the existing one-dimensional results to the current setup in a Hausdorff space. The existing results for R^n or measure metric spaces (in which $\Omega(t)$ are balls, see [1, 3, 17–19]) follow from ours, as well as from [23] and [15].

Unlike [23] and [15], our approach is elementary and does not require any advanced measure theory beyond σ-additivity. Note that binary partitions were used to prove sufficiency for the Hardy operator in the one-dimensional case in [2]. In contrast to [2], we avoid their auxiliary functions Φ and Φ_1 and apply discretization both for the upper and lower bounds in terms of the same functional of the weights.

The study of the approximation numbers (a-numbers) of the Hardy operator in the Lebesgue spaces on the half-line for parameters satisfying $1 < p \leq q < \infty$ started with the papers by Edmunds et al. [5, 6]. Next Edmunds and Stepanov [4] obtained the bounds for singular numbers of the Hardy-type operator with a polynomial kernel acting in the spaces $L^2(\mathbb{R}^+)$. Those results were extended by Lomakina and Stepanov [13] to the case $1 < p, q < \infty$; furthermore, two-sided bounds for the Schatten–von-Neumann norm were proved. However, in the case $1 < q < p < \infty$ their upper bound for the a-numbers depended on the index N of an approximation number. In this paper for $1 < q < p < \infty$ we state an upper bound that does not depend on N.

Full proofs of these results will be provided in a separate publication.

6.2 Hardy Operator Boundedness and Compactness

We write $A \asymp B$ to indicate that $c_1 A \le B \le c_2 A$ with constants c_1, c_2, independent of weights and measures.

Assumption 1 Let Ω be an open subset of a Hausdorff topological space X with σ-additive measures μ and ν. The measures are defined on a σ-algebra $\mathfrak{M}$ that contains the Borel-measurable sets. The weights u, v are assumed to be positive and finite almost everywhere.

Assumption 2

(a) $\{\Omega(t) : t \ge 0\}$ is a one-parameter family of open subsets of Ω which satisfy monotonicity: for $t_1 < t_2,\ \Omega(t_1)$ is a proper subset of $\Omega(t_2)$.
(b) The subsets $\Omega(t)$ start at the empty set and eventually cover almost all Ω: $\Omega(0) = \cap_{t>0}\Omega(t) = \emptyset$, $\nu\left(\Omega \backslash \cup_{t>0} \Omega(t)\right) = 0$.
(c) Further, denote $\omega(t) = \overline{\Omega(t)} \cap \overline{(\Omega\backslash\Omega(t))}$ the boundary of $\Omega(t)$ in the relative topology. We require the boundaries to be disjoint and cover almost all Ω:

$$\omega(t_1) \cap \omega(t_2) = \emptyset,\ t_1 \ne t_2,\ \nu(\Omega\backslash \cup_{t>0} \omega(t)) = 0. \tag{6.1}$$

(d) Passing to a different parametrization, if necessary, we can assume that

$$\nu\left(\Omega\backslash \cup_{t\le N} \omega(t)\right) > 0 \text{ for any } N < \infty. \tag{6.2}$$

(e) Finally, we assume that boundaries are thin in the sense that

$$\nu(\omega(t)) = 0 \text{ for all } t > 0. \tag{6.3}$$

This assumption has simple implications.

1. Equation (6.1) implies that for ν-almost each $y \in \Omega$ there exists a unique $\tau(y) > 0$ such that $y \in \omega(\tau(y))$, which allows us to define

$$Tf(y) = \int_{\Omega(\tau(y))} f d\nu,\ \ y \in \Omega, \tag{6.4}$$

for any non-negative $\mathfrak{M}$-measurable f. On the set $\Omega_0 \subset \Omega$ of those y for which $\tau(y)$ is not defined we can put $\tau(\Omega_0) = \emptyset$. (A more general definition of a Hardy-type operator is given in [7], which is more challenging to work with sets called below slices.)
2. Equation (6.2) and the fact that $\omega(t) \ne \emptyset,\ t > 0$, lead to $\tau(\Omega) = (0, \infty)$.
3. Because of (6.3) $\int_{\Omega(t)} f d\nu = \int_{\overline{\Omega(t)}} f d\nu$ and up to a set of ν-measure zero $\{x \in \Omega : \tau(x) > \tau(y)\} = \Omega\backslash\Omega(\tau(y))$.

For $0 \le a < b \le \infty$, we denote $\Omega([a, b]) = \Omega(b) \setminus \Omega(a)$.

$L^p_{vd\nu}(\Omega)$ denotes the space with the norm $\|f\|_{L^p_{vd\nu}(\Omega)} = \left(\int_\Omega |f|^p vd\nu\right)^{1/p}$, where v is a weight function. $\|T\| = \|T\|_{L^p_{vd\nu}(\Omega)\to L^q_{ud\mu}(\Omega)}$ is the norm of a linear operator T acting from $L^p_{vd\nu}(\Omega)$ to $L^q_{ud\mu}(\Omega)$:

$$\left[\int_\Omega \left|\int_{\Omega(\tau(x))} f d\nu\right|^q u(x)d\mu(x)\right]^{1/q} \le \|T\| \left(\int_\Omega |f|^p \, vd\nu\right)^{1/p}.$$

Denote

$$\Psi(t) = \left(\int_{\Omega\setminus\Omega(t)} ud\mu\right)^{1/q} \left(\int_{\Omega(t)} v^{-p'/p} d\nu\right)^{1/p'}.$$

Theorem 6.1 *If* $1 < p \le q < \infty$, *then* (6.4) *is bounded if and only if* $A < \infty$, *where* $A = \sup_{t>0} \Psi(t)$. *Besides,* $A \le \|T\| \le 4A$.

Let $0 < q < p$, $1 < p < \infty$ and put $1/r = 1/q - 1/p$,

$$\Phi(y) = \left(\int_{\Omega\setminus\Omega(\tau(y))} ud\mu\right)^{1/p} \left(\int_{\Omega(\tau(y))} v^{-p'/p} d\nu\right)^{1/p'}.$$

For $\Omega = (0, \infty)$, [14] and [24] have shown that $c_1 \|\Phi\|_{L^r_{ud\mu}(\Omega)} \le \|T\| \le c_2 \|\Phi\|_{L^r_{ud\mu}(\Omega)}$ with constants c_1, c_2 that depend on p, q and don't depend on the weights and measures.

Theorem 6.2 *If* $1 < p < \infty$ *and* $0 < q < p$, *then* (6.4) *is bounded if and only if* $B < \infty$, *where* $B = \left(\int_\Omega \Phi^r ud\mu\right)^{1/r}$. *Moreover,* $\|T\| \asymp B$.

With simple changes in the proofs, analogues of Theorems 6.1 and 6.2 hold for the adjoint operator $T^* : L^p_{vd\nu}(\Omega) \to L^q_{ud\mu}(\Omega)$,

$$T^* f\,(y) = \int_{\Omega\setminus\Omega(y)} f d\nu.$$

The next subject is compactness of T. Denote

$$a\,(x) = \int_{\Omega\setminus\Omega(x)} ud\mu,\ b\,(x) = \int_{\Omega(x)} v^{-p'/p} d\nu,\ 0 < x < \infty,$$

$$l_i = \limsup_{x\to i} a\,(x)^{1/q}\, b\,(x)^{1/p'},\ \text{for } i = 0, \infty,\ l = \max\{l_0, l_\infty\}.$$

Theorem 6.3 *(a) If* $1 < p \le q < \infty$, *then T is compact if and only if* $A < \infty$ *and* $l = 0$. *b) If* $1 < q < p < \infty$ *and T is bounded, then T is compact.*

6.3 Bounds for Approximation Numbers

Our next task is to obtain bounds for approximation numbers (a-numbers) of the operator (6.4). Let X and Y be two Banach spaces. For a bounded linear operator $T : X \to Y$ its n-th a-number, $n \in N$, is defined by

$$a_n(T) = \inf\{\|T - P\| : P : X \to Y \text{ is a bounded linear operator and } rank P < n\}.$$

For $[a, b] \subseteq [0, \infty)$, we initially consider the question of how well the operator $\chi_{[a,b]}T$ is approximated by averages. To this end, successively define

$$\mu_u(\Omega[a,b]) = \int_{\Omega[a,b]} u d\mu, \ \bar{T}f = \frac{1}{\mu_u(\Omega[a,b])}\int_{\Omega[a,b]}(Tf)u d\mu,$$
$$T_{[a,b]}f(x) = \chi_{\Omega[a,b]}(x)\left(Tf(x) - \bar{T}_{[a,b]}f\right).$$

Theorem 6.4 *Choose the point c so that $\mu_u(\Omega[a,c]) = \mu_u(\Omega[c,b]) = \frac{1}{2}\mu_u(\Omega[a,b])$. Let $1 < p \le q < \infty$, and denote*

$$A^*[a,c] = \sup_{a<\tau(x)<c}\left(\int_{\Omega[a,\tau(x)]} u d\mu\right)^{1/q}\left(\int_{\Omega[\tau(x),c]} v^{-p'/p} d\nu\right)^{1/p'},$$
$$A[c,b] = \sup_{c<\tau(x)<b}\left(\int_{\Omega[\tau(x),b]} u d\mu\right)^{1/q}\left(\int_{\Omega[c,\tau(x)]} v^{-p'/p} d\nu\right)^{1/p'}.$$

Then $\left\|T_{[a,b]}\right\|_{L^p_{vd\nu}(\Omega)\to L^q_{ud\mu}(\Omega)} \asymp \max\{A^*[a,c], A[c,b]\}$.

Denote $\mathbf{A}[a,b] = \max\{A^*[a,c], A[c,b]\}$, $0 \le a < b \le \infty$. Obviously, for any $0 < x < \infty$, we have $\mathbf{A}[a,b] \to 0$ if $a, b \to x$; $\mathbf{A}[a,b] > 0$ if $a < b$.

Throughout the following discussion, we make the crucial assumption that T is a compact operator.

Lemma 6.1 *Let $1 < p \le q < \infty$ and $0 < \varepsilon < \max\Psi$. There exist $0 = t_0 < t_1 < \ldots < t_N < t_{N+1} = \infty$ such that with the notation $\Delta_k = [t_k, t_{k+1})$, $k = 0, \ldots, N$ one has*

$$\sup_{t\in\Delta_0}\Psi(t) = \varepsilon, \ \max_{k=1,\ldots,N-2}\mathbf{A}(\Delta_k) = \varepsilon, \ \mathbf{A}(\Delta_{N-1}) \le \varepsilon, \ \sup_{t\in\Delta_N}\Psi(t) = \varepsilon. \tag{6.5}$$

We use the approach developed in [5].

Theorem 6.5 *Let $1 < p \le q < \infty$ and suppose the covering $\{\Omega_k : k = 0, \ldots, N\}$ satisfies (6.5). Then*

$$c_1\varepsilon(N-2)^{1/q-1/p} \le a_{N-1}(T), \ a_{N+1}(T) \le c_2\varepsilon. \tag{6.6}$$

Remark Obviously, when $p = q$, (6.6) gives a same-order two-sided bound for a-numbers. Besides, the upper bound on a-numbers gives an upper bound for the Gelfand, Kolmogorov and entropy numbers because the a-numbers are the largest among s-numbers of linear operators [16].

To consider the case $1 < q < p < \infty$, we assume that $\|T\| < \infty$ and therefore $B < \infty$ by Theorem 6.2. Denote

$$\Phi^*_{[a,b]}(x) = \left(\int_{\Omega[a,\tau(x)]} u d\mu\right)^{1/p} \left(\int_{\Omega[\tau(x),b]} v^{-p'/p} d\nu\right)^{1/p'},$$

$$\Phi_{[a,b]}(x) = \left(\int_{\Omega[\tau(x),b]} u d\mu\right)^{1/p} \left(\int_{\Omega[a,\tau(x)]} v^{-p'/p} d\nu\right)^{1/p'},$$

$$\mathbf{B}[a,b] = \left[\int_{\Omega[a,b]} \left(\Phi^*_{[a,c]}\chi_{[a,c]} + \Phi_{[c,b]}\chi_{[c,b]}\right)^r u d\mu\right]^{1/r},$$

where $1/r = 1/q - 1/p$ and $c = c(a,b)$ is the constant from Theorem 6.4.

Bound from Above Let $0 < \varepsilon < B$. Select t', t'' to satisfy

$$\left(\int_{\Omega(t')} \Phi^r u d\mu\right)^{1/r} = \varepsilon, \ \left(\int_{\Omega[t'',\infty]} \Phi^r u d\mu\right)^{1/r} = \varepsilon. \tag{6.7}$$

Let $\{\Delta_k : k = 1, \ldots, N\}$ be a uniform and finite partition of $\left[t', t''\right]$ into segments Δ_k of length m. The number m can be chosen so that

$$\left(\sum_{k=1}^{N} \mathbf{B}(\Delta_k)^r\right)^{1/r} = \varepsilon.$$

With the partition defined above, we have the following theorem:

Theorem 6.6 *Suppose* $1 < q < p < \infty$, T *is bounded and* $0 < \varepsilon < B$. *Then* $a_{N+1}(T) \le c\varepsilon$.

Bound from Below Let t', t'' be chosen as in (6.7) and put $t_0 = 0$, $t_1 = t'$. On the n-th step, if $\sup_{t>t_n} \mathbf{B}(t_n, t) \ge \varepsilon$ then we put $t_{n+1} = \min\{t > t_n : \mathbf{B}(t_n, t) = \varepsilon\}$. If $\sup_{t>t_n} \mathbf{B}(t_n, t) < \varepsilon$ we put $t_{n+1} = \infty$. It is easy to show that this process stops in a finite number of steps.

With the partition we have defined here we can state the lower bound.

Theorem 6.7 *Suppose* $1 < q < p < \infty$, T *is bounded and* $0 < \varepsilon < B$. *Then* $a_{N-1}(T) \ge c\varepsilon$.

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References

1. Bandaliev, R.A.: Application of multidimensional Hardy operator and its connection with a certain nonlinear differential equation in weighted variable Lebesgue spaces. Ann. Funct. Anal. **4**(2), 118–130 (2013)
2. Bernardis, A., Martín-Reyes, F., Salvador, P.: A new proof of the characterization of the weighted Hardy inequality. Proc. R. Soc. Edinburgh Sect A Math. **135**(5), 941–945 (2005)
3. Drábek, P., Heinig, H.P., Kufner, A.: Higher-dimensional Hardy inequality. Gen. Inequal. **7**, 3–16 (1997)
4. Edmunds, D.E., Stepanov, V.D.: On the singular numbers of certain Volterra integral operators. J. Funct. Anal. **134**(1), 222–246 (1995)
5. Edmunds, D.E., Evans, W.D., Harris, D.J.: Approximation numbers of certain Volterra integral operators. J. Lond. Math. Soc. **37**(2), 471–489 (1988)
6. Edmunds, D.E., Evans, W.D., Harris, D.J.: Two-sided estimates of the approximation numbers of certain Volterra inteqral operators. Stud. Math. **124**(1), 59–80 (1997)
7. Edmunds, D.E., Kokilashvili, V., Meskhi, A.: Bounded and Compact Integral Operators. Mathematics and its Applications, vol. 543. Kluwer, Dordrecht (2002)
8. Heinig, H.P., Sinnamon, G.: Mapping properties of integral averaging operators. Stud. Math. **129**(2), 157–177 (1998)
9. Kufner, A., Opic, B.: Hardy-Type Inequalities. Pitman Research Notes in Mathematics Series, vol. 219. Longman Scientific & Technical, Harlow (1990)
10. Kufner, A., Persson, L.E.: Weighted Inequalities of Hardy Type. World Scientific, River Edge (2003)
11. Kufner, A., Maligranda, L., Persson, L.E.: The Hardy Inequality. About its History and Some Related Results. Vydavatelský Servis, Plzeň (2007)
12. Kufner, A., Persson, L.E., Samko, N.: Weighted Inequalities of Hardy Type, 2nd edn. World Scientific, Hackensack (2017)
13. Lomakina, E.N., Stepanov, V.D.: On Asymptotic Behaviour of the Approximation Numbers and Estimates of Schatten von Neumann Norms of the Hardy–Type Integral Operators. Function Spaces and Application. Narosa Publishing Hause, New Delhi, pp. 153–187 (2000)
14. Maz'ya, V.: Sobolev Spaces with Applications to Elliptic Partial Differential Equations. Fundamental Principles of Mathematical Sciences, vol. 342. Springer, Heidelberg (2011)
15. Mynbaev, K.T.: Three weight Hardy inequality on measure topological spaces. Eurasian Math. J. **14**(2), 58–78 (2023)
16. Pietsch, A.: s-Numbers of operators in Banach spaces. Stud. Math. **5**(3), 201–223 (1974)
17. Ruzhansky, M., Verma, D.: Hardy inequalities on metric measure spaces. Proc. R. Soc. A. **475**(2223), 20180310, 15pp. (2019)
18. Ruzhansky, M., Verma, D.: Hardy inequalities on metric measure spaces, II: the case $p > q$. Proc. R. Soc. A. **477**(2250), 20210136, 16pp. (2021)
19. Ruzhansky, M., Yessirkegenov, N.: Hardy, Hardy-Sobolev, Hardy-Littlewood-Sobolev and Caffarelli-Kohn-Nirenberg inequalities on general Lie groups. arXiv:1810.08845v2 [math.FA] (2019)
20. Sawyer, E.: Weighted inequalities for the two-dimensional Hardy operator. Stud. Math. **82**(1), 1–16 (1985)
21. Sawyer, E.: Boundedness of classical operators on classical Lorentz spaces. Stud. Math. **96**(2), 145–158 (1990)
22. Sinnamon, G.: One-dimensional Hardy-type inequalities in many dimensions. Proc. Roy. Soc. Edinburgh Sect. A **128**(4), 833–848 (1998)

23. Sinnamon, G.: Hardy inequalities in normal form. Trans. Am. Math. Soc. **375**(2), 961–995 (2022)
24. Sinnamon, G., Stepanov, V.D.: The weighted Hardy inequality: new proofs and the case $p = 1$. J. Lond. Math. Soc. **54**(1), 89–101 (1996)
25. Stepanov, V.D.: The weighted Hardy's inequality for non-increasing functions. Trans. Am. Math. Soc. **338**(1), 173–186 (1993)
26. Stepanov, V.D., Shambilova, G.E.: On two-dimensional bilinear inequalities with rectangular Hardy operators in weighted Lebesgue spaces. Proc. Steklov Inst. Math. **312**, 241–248 (2021)
27. Wedestig, A.: Weighted Inequalities of Hardy type and their Limiting Inequalities. Doctoral thesis, Department of Mathematics, Lulea University of Technology, Sweden (2003)

Part II
Theory of Applied Mathematics

Chapter 7
A Numerical Algorithm for the Third Order Delay Partial Differential Equation with Robin Boundary Condition

Suleiman Ibrahim and Deniz Agirseven

Abstract In the present paper, the initial value problem for the third order delay partial differential equation with Robin boundary condition is investigated. The first order of accuracy difference scheme for the numerical solution of the third order delay partial differential equation with Robin boundary condition is presented. The illustrative numerical results are provided.

7.1 Introduction

Over the years, local and nonlocal boundary value problems for third order partial differential equations have been studied (for instance, see [1–8]). In many engineering applications, time delay is one of the most common phenomena that frequently occurs. In control theory, the process of sampled-data control is a typical example where time delay happens in the transmission from measurement to controller. Theory and applications of delay linear and nonlinear third order ordinary differential and difference equations were widely investigated (see, e.g., [9–18], and the references given therein).

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7.2 The Third Order Delay Partial Differential Equation with Robin Boundary Condition

We consider the initial boundary value problem for the third order partial differential equation with time delay

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - (a(x)u_{tx}(t,x))_x + \delta u_t(t,x) \\ = b\left(-(a(x)u_x(t-w,x))_x + \delta u(t-w,x)\right) + f(t,x), \\ 0 < t < \infty,\ 0 < x < l, \\ u(t,x) = g(t,x), -w \le t \le 0, 0 \le x \le l, \\ u(t,0) = \alpha u_x(t,0),\ -u(t,l) = \beta u_x(t,l),\quad 0 \le t < \infty. \end{cases} \tag{7.1}$$

Under compatibility conditions, problem (7.1) has a unique solution $u(t,x)$ for the smooth functions $a(x) \ge a > 0,\ x \in (0,l),\ \delta > 0, \alpha, \beta \ge 0,\ g(t,x),\ -w \le t \le 0,\ 0 \le x \le l,\ f(t,x),\ 0 < t < \infty,\ 0 < x < l$, and $b \in R^1$. The discretization of problem (7.1) is carried out in two steps. In the first step, we define the grid space

$$[0,l]_h = \{x = x_n : x_n = nh,\ 0 \le n \le M,\ Mh = l\}.$$

Let us introduce the Hilbert space $L_{2h} = L_2([0,l]_h)$ of the grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ defined on $[0,l]_h$, equipped with the norm

$$\|\varphi^h\|_{L_{2h}} = \left(\sum_{x\in[0,l]_h} |\varphi(x)|^2 h\right)^{1/2}.$$

To the differential operator A^x defined by (7.1), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\overline{x}})_{x,n} + \delta\varphi_n\}_1^{M-1} \tag{7.2}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ satisfying the conditions $(h+\alpha)\,\varphi_0 = \alpha\varphi_1$, $(h+\beta)\,\varphi_M = \beta\varphi_{M-1}$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we reach the initial value problem for an infinite system of ordinary differential equations

$$\begin{cases} \frac{d^3u^h(t,x)}{dt^3} + A_h^x \frac{du^h(t,x)}{dt} = bA_h^x u(t-w,x) + f^h(t,x),\ 0 < t < \infty, x \in [0,l]_h, \\ u^h(t,x) = g^h(t,x), -w \le t \le 0, x \in [0,l]_h. \end{cases} \tag{7.3}$$

In the second step, we replace (7.3) with the difference scheme of the first order of accuracy presented in [19]. We get

$$\begin{cases} \frac{u_{k+2}^h(x)-3u_{k+1}^h(x)+3u_k^h(x)-u_{k-1}^h(x)}{\tau^3} + A_h^x \frac{u_{k+2}^h(x)-u_{k+1}^h(x)}{\tau} \\ = bA_h^x u_{k-N}^h(-x) + f_k^h(x),\ f_k^h(x) = f^h(t_k,x), k \ge 1,\ x \in [0,l]_h, \\ u_k^h(x) = g^h(t_k,x), -N \le k \le 0, \\ (I_h + \tau^2 A_h^x)\frac{u_1^h(x)-u_0^h(x)}{\tau} = g_t^h(0,x), \\ (I_h + \tau^2 A_h^x)\frac{u_2^h(x)-2u_1^h(x)+u_0^h(x)}{\tau^2} = g_{tt}^h(0,x), x \in [0,l]_h, \\ (I_h + \tau^2 A_h^x)\frac{u_{mN+1}^h(x)-u_{mN}^h(x)}{\tau} = \frac{u_{mN}^h(x)-u_{mN-1}^h(x)}{\tau}, \\ (I_h + \tau^2 A_h^x)\frac{u_{mN+2}^h(x)-2u_{mN+1}^h(x)+u_{mN}^h(x)}{\tau^2} \\ = \frac{u_{mN}^h(x)-2u_{mN-1}^h(x)+u_{mN-2}^h(x)}{\tau^2}, m = 1, 2, \dots . \end{cases} \tag{7.4}$$

Theorem 7.1 *For the solution of difference scheme (7.4), the following stability estimates*

$$\max\left\{\|D_\tau^2 u_k^h\|_{C_\tau(W_{2h}^1)}, \|D_\tau^1 u_k^h\|_{C_\tau(W_{2h}^2)}, \frac{1}{2}\|u_k^h\|_{C_\tau(W_{2h}^3)}\right\}$$

$$\leq C_1\left[(2+\tau|b|(N-2))^m b_0^h + \sum_{j=1}^{m}(2+\tau|b|(N-2))^{m-j}\tau\right.$$

$$\left.\times\sum_{s=(j-1)N+1}^{jN}\|A^{\frac{1}{2}}f(t_s)\|_{W_{2h}^1}\right], m=0,1,\dots,$$

$$b_0^h=\max\left\{\max_{-N\leq k\leq 0}\|A^{\frac{1}{2}}g_{tt}^h(t_k)\|_{W_{2h}^1}, \max_{-N\leq k\leq 0}\|g_t^h(t_k)\|_{W_{2h}^2}, \max_{-N\leq k\leq 0}\|g^h(t_k)\|_{W_{2h}^3}\right\},$$

$$\|D_\tau^2 u_k^h\|_{C_\tau(W_{2h}^1)}=\max_{0\leq k\leq(m+1)N-2}\left\|\frac{u_{k+2}^h-2u_{k+1}^h+u_k^h}{\tau^2}\right\|_{W_{2h}^1},$$

$$\|D_\tau^1 u_k^h\|_{C_\tau(W_{2h}^2)}=\max_{1\leq k\leq(m+1)N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{W_{2h}^2},$$

$$\|u_k^h\|_{C_\tau(W_{2h}^3)}=\max_{0\leq k\leq(m+1)N}\|u_k^h\|_{W_{2h}^3}$$

hold, where C_1 does not depend on $g^h(t_k)$, and $f_k^h(x)$.

Proof Difference scheme (7.4) can be written in abstract form

$$\begin{cases}\frac{u_{k+2}^h-3u_{k+1}^h+3u_k^h-u_{k-1}^h}{\tau^3}+A_h\frac{u_{k+2}^h-u_{k+1}^h}{\tau}=bA_hu_{k-N}^h+f_k^h, k\geq 1,\\ u_k^h=g_k^h, -N\leq k\leq 0,\\ (I_h+\tau^2A_h)\frac{u_1^h-u_0^h}{\tau}=g_t^h(0), (I_h+\tau^2A_h)\frac{u_2^h-2u_1^h+u_0^h}{\tau^2}=g_{tt}^h(0),\\ (I_h+\tau^2A_h)\frac{u_{mN+1}^h-u_{mN}^h}{\tau}=\frac{u_{mN}^h-u_{mN-1}^h}{\tau},\\ (I_h+\tau^2A_h)\frac{u_{mN+2}^h-2u_{mN+1}^h+u_{mN}^h}{\tau^2}=\frac{u_{mN}^h-2u_{mN-1}^h+u_{mN-2}^h}{\tau^2}, m=1,2,\dots\end{cases}\tag{7.5}$$

in a Hilbert space L_{2h} with self-adjoint positive definite operator $A_h = A_h^x$ by formula (7.2). Here, $g_k^h = g_k^h(x)$, $f_k^h = f_k^h(x)$ and $u_k^h = u_k^h(x)$ are known and unknown abstract mesh functions respectively defined on $[0,l]_h$ with the values in $H = L_{2h}$. Therefore, estimate of Theorem 7.1 follows from estimate of abstract Theorem 2 [19]. Theorem 7.1 is proved. □

7.3 Numerical Algorithm for the Third Order Delay Partial Differential Equation

We present the algorithm for the numerical solution of the initial boundary value problem for third order delay partial differential equation with Robin boundary condition

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} - \frac{\partial^3 u(t,x)}{\partial t \partial x^2} + 8\frac{\partial u(t,x)}{\partial t} = -0.1(-\frac{\partial^2 u(t-1,x)}{\partial x^2} + 8u(t-1,x)) \\ -32e^{-2t}\cos 2x + 1.2e^{-2(t-1)}\cos 2x, \\ 0 < t < \infty,\ -\pi < x < \pi, \\ u(t,x) = e^{-2t}\cos 2x,\ -1 \le t \le 0,\ 0 \le x \le \pi, \\ u(t,0) - e^{-2t} = 28u_x(t,0), -u(t,\pi) + e^{-2t} = 28u_x(t,\pi),\ 0 \le t < \infty. \end{cases} \tag{7.6}$$

The exact solution of problem (7.6) is

$$u(t,x) = e^{-2t}\cos 2x,\ 0 \le x \le \pi, -1 \le t < \infty.$$

For the approximate solutions of problem (7.6), using the set of grid points

$$[-1,\infty)_\tau \times [\,0,\pi]_h$$

$$= \{(t_k, x_n) : t_k = k\tau, k \ge -N, N\tau = 1, x_n = nh,\ 0 \le n \le M, Mh = \pi\},$$

we get the first order of accuracy difference scheme in t

$$\begin{cases} \frac{u_n^{k+2}-3u_n^{k+1}+3u_n^k-u_n^{k-1}}{\tau^3} - \frac{u_{n+1}^{k+2}-u_{n+1}^{k+1}-2(u_n^{k+2}-u_n^{k+1})+u_{n-1}^{k+2}-u_{n-1}^{k+1}}{\tau h^2} \\ +8\frac{u_n^{k+2}-u_n^{k+1}}{\tau} = -(0.1)\left(-\frac{u_{n+1}^{k-N}-2u_n^{k-N}+u_{n-1}^{k-N}}{h^2} + 8u_n^{k-N}\right) \\ -32e^{-2t_k}\cos 2x_n + 1.2e^{-2(t_{k-N})}\cos 2x_n, \\ t_k = k\tau,\ mN+1 \le k \le (m+1)N-2, \\ m = 0, 1, \dots,\ 1 \le n \le M-1, \\ N\tau = 1,\ x_n = nh,\ 1 \le n \le M-1,\ Mh = \pi, \\ u_n^0 = \cos(2nh), \\ \frac{u_n^1-u_n^0}{\tau} + \tau(-\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2} + 8u_n^1) + \\ \tau(\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2} - 8u_n^0) = -2\cos(2nh), \\ \frac{u_n^2-2u_n^1+u_n^0}{\tau^2} + (-\frac{u_{n+1}^2-2u_n^2+u_{n-1}^2}{h^2} + 8u_n^2) \\ +2(\frac{u_{n+1}^1-2u_n^1+u_{n-1}^1}{h^2} - 8u_n^1) \\ +(-\frac{u_{n+1}^0-2u_n^0+u_{n-1}^0}{h^2} + 8u_n^0) = 4\cos(2nh),\ 1 \le n \le M-1, \\ \frac{u_n^{mN+1}-u_n^{mN}}{\tau} + \tau(-\frac{u_{n+1}^{mN+1}-2u_n^{mN+1}+u_{n-1}^{mN+1}}{h^2} + 8u_n^{mN+1}) \\ +\tau(\frac{u_{n+1}^{mN}-2u_n^{mN}+u_{n-1}^{mN}}{h^2} - 8u_n^{mN}) = \frac{u_n^{mN}-u_n^{mN-1}}{\tau}, \\ \frac{u_n^{mN+2}-2u_n^{mN+1}+u_n^{mN}}{\tau^2} \\ +(-\frac{u_{n+1}^{mN+2}-2u_n^{mN+2}+u_{n-1}^{mN+2}}{h^2} + 8u_n^{mN+2}) \\ +2(\frac{u_{n+1}^{mN+1}-2u_n^{mN+1}+u_{n-1}^{mN+1}}{h^2} - 8u_n^{mN+1}) \\ +(-\frac{u_{n+1}^{mN}-2u_n^{mN}+u_{n-1}^{mN}}{h^2} + 8u_n^{mN}) \\ = \frac{u_n^{mN}-2u_n^{mN-1}+u_n^{mN-2}}{\tau^2},\ 1 \le n \le M-1,\ m = 1, 2, \dots, \\ u_0^k - e^{-2t_k} = \frac{28}{h}\left(u_1^k - u_0^k\right), \\ -u_M^k + e^{-2t_k} = \frac{28}{h}\left(u_M^k - u_{M-1}^k\right),\ 0 \le k < \infty, \\ mN \le k \le (m+1)N,\ m = 1, 2, \dots . \end{cases} \tag{7.7}$$

We can write (7.7) in the matrix form as follows:

$$\begin{cases} BU^{k+2}+CU^{k+1}+DU^{k}+EU^{k-1}=\varphi(U^{k-N}),\, k=1,2,3,\ldots, \\ U^{0}=\begin{bmatrix} \cos\,(2(0)h) \\ \cos\,(2(1)h) \\ \vdots \\ \cos\,(2(M-1)h) \\ \cos\,(2(M)h) \end{bmatrix}, \\ U^{1}=LHU^{0}, \\ U^{2}=YPU^{1}+YQU^{0}, \\ U^{mN+1}=LJU^{mN}+LWU^{mN-1}, \\ U^{mN+2}=YPU^{mN+1}+YXU^{mN}+YSU^{mN-1} \\ +YZU^{mN-2},\, m=1,2,\ldots, \end{cases} \tag{7.8}$$

where B, C, D, E, F, H, J, P, Q, S, V, W, X and Z are $(M+1)\times(M+1)$ matrices, $\varphi(U^{k-N})$, U^{0}, U^{1} and $U^{r}, r=k, k\pm 1, k+2$ are $(M+1)\times 1$ column vectors defined by

$$B=\begin{bmatrix} 1+\frac{28}{h} & -\frac{28}{h} & 0\;0\cdot 0 & 0 & 0 \\ a & b & c\quad\cdot 0 & 0 & 0 \\ 0 & a & b\;c\cdot 0 & 0 & 0 \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot & \cdot \\ 0 & 0 & 0\;0\cdot b & c & 0 \\ 0 & 0 & 0\;0\cdot a & b & c \\ 0 & 0 & 0\;0\cdot 0 & -\frac{28}{h} & 1+\frac{28}{h} \end{bmatrix}, C=\begin{bmatrix} 0\;0\;0\;0\cdot 0\;0\;0 \\ l\;c\;l\quad\cdot 0\;0\;0 \\ 0\;l\;c\;l\cdot 0\;0\;0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ 0\;0\;0\;0\cdot c\;l\;0 \\ 0\;0\;0\;0\cdot l\;c\;l \\ 0\;0\;0\;0\cdot 0\;0\;0 \end{bmatrix},$$

$$D=\begin{bmatrix} 0\;0\;0\;0\cdot 0\;0\;0 \\ 0\;d\;0\quad\cdot 0\;0\;0 \\ 0\;0\;d\;0\cdot 0\;0\;0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ 0\;0\;0\;0\cdot d\;0\;0 \\ 0\;0\;0\;0\cdot 0\;d\;0 \\ 0\;0\;0\;0\cdot 0\;0\;0 \end{bmatrix}, E=\begin{bmatrix} 0\;0\;0\;0\cdot 0\;0\;0 \\ 0\;e\;0\quad\cdot 0\;0\;0 \\ 0\;0\;e\;0\cdot 0\;0\;0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ 0\;0\;0\;0\cdot e\;0\;0 \\ 0\;0\;0\;0\cdot 0\;e\;0 \\ 0\;0\;0\;0\cdot 0\;0\;0 \end{bmatrix},$$

$$W=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\0&w&0&&\cdot&0&0&0\\0&0&w&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&w&0&0\\0&0&0&0&\cdot&0&w&0\\0&0&0&0&\cdot&0&0&0\end{bmatrix},S=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\0&s&0&&\cdot&0&0&0\\0&0&s&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&s&0&0\\0&0&0&0&\cdot&0&s&0\\0&0&0&0&\cdot&0&0&0\end{bmatrix},$$

$$Z=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\0&z&0&&\cdot&0&0&0\\0&0&z&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&z&0&0\\0&0&0&0&\cdot&0&z&0\\0&0&0&0&\cdot&0&0&0\end{bmatrix},J=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\j^*&j&j^*&&\cdot&0&0&0\\0&j^*&j&j^*&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&j&j^*&0\\0&0&0&0&\cdot&j^*&j&j^*\\0&0&0&0&\cdot&0&0&0\end{bmatrix},$$

$$F=\begin{bmatrix}1+\frac{28}{h}&-\frac{28}{h}&0&0&\cdot&0&0&0\\f^*&f&f^*&&\cdot&0&0&0\\0&f^*&f&f^*&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&f&f^*&0\\0&0&0&0&\cdot&f^*&f&f^*\\0&0&0&0&\cdot&0&-\frac{28}{h}&1+\frac{28}{h}\end{bmatrix},$$

$$P=\begin{bmatrix}0&0&0&0&\cdot&0&0&0\\p^*&p&p^*&&\cdot&0&0&0\\0&p^*&p&p^*&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&p&p^*&0\\0&0&0&0&\cdot&p^*&p&p^*\\0&0&0&0&\cdot&0&0&0\end{bmatrix},$$

$$V=\begin{bmatrix}1+\frac{28}{h}&-\frac{28}{h}&0&0&\cdot&0&0&0\\v*&v&v*&&\cdot&0&0&0\\0&v*&v&v*&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&0&\cdot&v&v*&0\\0&0&0&0&\cdot&v*&v&v*\\0&0&0&0&\cdot&0&-\frac{28}{h}&1+\frac{28}{h}\end{bmatrix},$$

$$H=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ h^* & e^* & h^* & & \cdot & 0 & 0 & 0 \\ 0 & h^* & e^* & h^* & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & e^* & h^* & 0 \\ 0 & 0 & 0 & 0 & \cdot & h^* & e^* & h^* \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}, X=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ x^* & s^* & x^* & & \cdot & 0 & 0 & 0 \\ 0 & x^* & s^* & x^* & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & s^* & x^* & 0 \\ 0 & 0 & 0 & 0 & \cdot & x^* & s^* & x^* \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$Q=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ q^* & q & q^* & & \cdot & 0 & 0 & 0 \\ 0 & q^* & q & q^* & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & q & q^* & 0 \\ 0 & 0 & 0 & 0 & \cdot & q^* & q & q^* \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}, \varphi(U^{k-N})=\begin{bmatrix} \varphi_0^k \\ \varphi_1^k \\ \vdots \\ \varphi_{M-1}^k \\ \varphi_M^k \end{bmatrix}, \ U^r=\begin{bmatrix} U_0^r \\ U_{-M+1}^r \\ \vdots \\ U_{M-1}^r \\ U_M^r \end{bmatrix},$$

$r=k, k\pm 1, k+2$, where

$$\varphi_n^k=-(0.1)\left(-\frac{u_{n+1}^{k-N}-2u_n^{k-N}+u_{n-1}^{k-N}}{h^2}+8u_n^{k-N}\right)-32e^{-2t_k}\cos 2x_n$$

$$+1.2e^{-2(t_{k-N})}\cos 2x_n, t_k=k\tau,\ mN+1\le k\le (m+1)N-2,$$

$$m=0,1,\ldots,\ 1\le n\le M-1,$$

$a=-\frac{1}{\tau h^2}$, $b=\frac{1}{\tau^3}+\frac{2}{\tau h^2}+\frac{8}{\tau}$, $c=-\frac{3}{\tau^3}-\frac{2}{\tau h^2}-\frac{8}{\tau}$, $l=-a$, $d=\frac{3}{\tau^3}$, $e=-\frac{1}{\tau^3}$, $w=-\frac{1}{\tau}$, $s=-\frac{2}{\tau^2}$ $z=\frac{1}{\tau^2}$, $f=\frac{2\tau}{h^2}+\frac{1}{\tau}+8\tau$, $f^*=-\frac{\tau}{h^2}$, $p=\frac{2}{\tau^2}+\frac{4}{h^2}+16$, $p^*=-\frac{2}{h^2}$, $v=\frac{1}{2}p$, $v^*=\frac{1}{2}p^*$, $j=f+\frac{1}{\tau}$, $j^*=f^*$, $h^*=f^*, e^*=f-2$, $s^*=p^*-8$, $x^*=-v^*$, $q=-\frac{1}{\tau^2}-\frac{2}{h^2}-4$, $q^*=x^*$, $L=F^{-1}$, $Y=V^{-1}$.

7.4 Numerical Analysis

The numerical solutions are recorded for different values of N and M, and u_n^k represents the numerical solution of this difference scheme at $u(t_k, x_n)$. Table 7.1 is constructed for $N=M=30, 60, 120$ in $t\in[0,1]$, $t\in[1,2]$, $t\in[2,3]$ respectively and the errors are computed by

$$mE_M^N=\max_{mN+1\le k\le (m+1)N,\ -M\le n\le M}|u(t_k,x_n)-u_n^k|.$$

Table 7.1 Errors of difference scheme (7.7)

(N, M)	$N = M = 30$	$N = M = 60$	$N = M = 120$
$t \in [0, 1]$	0.1886	0.0987	0.0504
$t \in [1, 2]$	0.2462	0.1210	0.0601
$t \in [2, 3]$	0.1615	0.0741	0.0376

If N and M are doubled, the values of the errors are decreased by a factor of approximately $1/2$ for the first order of accuracy difference scheme (7.7). The errors presented in this table indicate the accuracy of difference scheme.

7.5 Conclusion

In the present paper, the first order of accuracy difference schemes for the numerical solution of the third order delay partial differential equation with Robin boundary condition are presented. The illustrative numerical results are provided.

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References

1. Amirov, S., Kozhanov, A.I.: A mixed problem for a class of strongly nonlinear higher-order equations of Sobolev type. Dokl. Math. **88**, 446–448 (2013)
2. Niu, J., Li, P.: Numerical algorithm for the third-order partial differential equation with threepoint boundary value problem. Abstr. Appl. Anal. **2014**, 1–11 (2014)
3. Apakov, Y.: On the solution of a boundary-value problem for a third-order equation with multiple characteristics. Ukrainian Math. J. **64**, 1–12 (2012)
4. Apakov, Y., Irgashev, B.: Boundary-value problem for a generate high-odd order equation. Ukrainian Math. J. **66**, 1475–1490 (2015)
5. Apakov, Y., Rutkauskas, S.: On a boundary value problem to third-order PDE with multiple characteristics. Nonlinear Anal. Model. Control **16**, 255–269 (2011)
6. Belakroum, K., Ashyralyev, A., Guezane-Lakoud, A.: A note on the nonlocal boundary value problem for a third order partial differential equation. Filomat **32**, 801–808 (2018)
7. Kudu, M., Amirali, I.: Method of lines for third order partial differential equations. J. Appl. Math. **2**, 33–36 (2014)
8. Latrous, C., Memou, A.: A three-point boundary value problem with an integral condition for a third-order partial differential equation. Abstr. Appl. Anal. **2005**, 1–9 (2005)
9. Cahlon, B., Schmidt, D.: Stability criteria for certain third-order delay differential equations. J. Comput. Appl. Math. **188**, 319–335 (2006)
10. Baculíková, B., Dzurina, J., Rogovchenko, Y.V.: Oscillation of third order trinomial delay differential equations. Appl. Math. Comput. **218**, 7023–7033 (2012)

11. Afuwape, A.U., Omeike, M.O.: Stability and boundedness of solutions of a kind of third-order delay differential equations. Comput. Appl. Math. **29**, 329–342 (2010)
12. Grace, S.R.: Oscillation criteria for third order nonlinear delay differential equations with damping. Opuscula Math. **35**, 485–497 (2015)
13. Xiang, H.: Oscillation of third order nonlinear neutral differential equations with distributed time delay. Ital. J. Pure Appl. Math. **36**, 769–782 (2016)
14. Sobolevskii, P.E.: Difference Methods for the Approximate Solution of Differential Equations. Voronezh State University Press, Voronezh, Russia (1975)
15. Ashyralyev, A., Hincal, E., Ibrahim S.: Stability of the third order partial differential equations with time delay. AIP Conf. Proc. **1997**, 020086 (2018)
16. Ashyralyev, A., Hincal, E., Ibrahim, S.: A numerical algorithm for the third-order partial differential equation with time delay. AIP Conf. Proc. **2183**, 070014 (2019)
17. Ashyralyev, A., Sarsenbi, A.M.: Well-posedness of an elliptic equation with involution. Electron. J. Differ. Equ. **2015**, 1–8 (2015)
18. Ashyralyev, A., Ibrahim, S., Hincal E.: On stability of the third order partial delay differential equation with nomlocal boundary conditions. Int. J. Appl. Math. **35(1)**, 1–14 (2022)
19. Ashyralyev, A., Hincal, E., Ibrahim S.: On the absolute stable difference scheme for third order delay partial differential equations. Symmetry **12(6)**, 1033, 23pp. (2020)

Chapter 8
Solution of the Problem of Generalized Localization for Spherical Partial Sums of Multiple Fourier Series

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Abstract It is well known that Luzin's conjecture has a positive solution in one dimensional case and it is still open in multidimensional case for the spherical partial sums: $S_\lambda f(x) = \sum_{|n|^2<\lambda} f_n e^{inx}$, where f_n are the Fourier coefficients of f. Historically, progress with solving the Luzin's conjecture has been made by considering easier problems. One of such easier problems for $S_\lambda f(x)$ was suggested by V. A. Il'in in 1968 and this problem is called the generalized localization principle for the spherical partial sums. In 1976, well-known specialists in this field raised the question of the validity of the principle of generalized localization at least for double trigonometric series. In this paper, we give a solution of this problem and indicate a sketch of the proof. In addition, a result is given on the convergence of spherical partial sums for functions from the Sobolev class.

8.1 Introduction

Let $\{f_n\}$, $n \in \mathbb{Z}^N$, be the Fourier coefficients of the function $f \in L_2(\mathbb{T}^N)$, $N \geq 2$, i.e.

$$f_n = (2\pi)^{-N} \int\limits_{\mathbb{T}^N} f(y) e^{-iny} dy,$$

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where $\mathbb{T}^N$ is an N-dimensional torus: $\mathbb{T}^N = (\pi, \pi]^N$. Spherical partial sums of the Fourier series are determined by the formula:

$$S_\lambda f(x) = \sum_{|n|^2<\lambda} f_n \, e^{inx}, \tag{8.1}$$

where $nx = n_1x_1 + n_2x_2 + \ldots + n_Nx_N$ and $|n| = \sqrt{n_1^2 + n_2^2 + \ldots + n_N^2}$.

One of the classical problems in the theory of Fourier series is the problem of pointwise convergence of $S_\lambda f(x)$ to f as $\lambda \to \infty$: for what classes of functions f are the partial sums $S_\lambda f(x)$ converge to f at a given point, almost everywhere, at all points, or uniformly.

In this paper, we will study the problem of convergence almost everywhere of the series $S_\lambda f(x)$, which is one of the most attractive problems in metric function theory. Natural classes for studying convergence almost everywhere are the spaces L_p, $1 \le p < \infty$.

It should be noted that even in the simplest case of $N = 1$ and $p = 2$ the problem of convergence almost everywhere was solved relatively recently, in 1966, by Carleson [1]. The corresponding hypothesis was expressed back in 1915 in his dissertation "Integrals and trigonometric series" by Lebesgue's student N.N. Luzin. Improving Carleson's method, German mathematician Hunt [2] extended this result to functions from the classes L_p, $p > 1$. If $p = 1$, then this is no longer true due to the well-known example of a function from the class L_1 with Fourier series diverging everywhere in $\mathbb{T}^1$, constructed by Kolmogorov in 1922.

For multidimensional Fourier series, the validity of the Carleson-Hunt result depends on the type of partial sums (see [3]). Thus, for square partial sums of N-fold Fourier series, Sjölin extended Hunt's theorem. However, as Fefferman showed, convergence over rectangles leads to completely new effects: there is a continuous function $f(x_1, x_2)$ whose Fourier series diverges indefinitely over rectangles at each interior point of the square $\mathbb{T}^2$.

As for spherical partial sums S_λ, attempts to transfer Luzin's conjecture undertaken by mathematicians such as L. Carleson and C. Fefferman, ultimately turned out to be unsuccessful and Luzin problem remains open to this day.

Historically, progress in solving Luzin's conjecture has been achieved by considering simpler problems. For example, we obtain a simpler problem if we impose on the decomposed function f some additional conditions that simplify the study of convergence almost everywhere. As such conditions, we can take the vanishing of the function f in some subdomain $G \subset \mathbb{T}^N$ and study the almost everywhere convergence of spherical means in G. If there is such convergence, then they say (in the terminology of V.A. Il'in [4]), that *the principle of generalized localization* is valid.

As an alternative way to simplify the situation, one can study the convergence of $S_\lambda f(x)$ not for functions from the entire class L_p, but for functions from the subspace $L_p(\mathbb{T}^N)$ (for example, from the Sobolev classes), which have some smoothness.

It should be emphasized that solving the principle of generalized localization is the first important step towards solving N.N. Luzin's problem for multidimensional Fourier series. That is why in 1976, well known specialists in this field raised the question of the validity of the principle of generalized localization at least for double trigonometric series (see [5]).

Many specialists began to solve this problem. As a result, through the efforts of mathematicians such as P. Sjölin, A. Carbery, F. Soria, J.L. Rubio de France, L. Vega, A. Bastys, the validity of the generalized localization principle was established for expansions into multiple trigonometric Fourier integrals (see [6–15]). But the case of a multiple Fourier series turned out to be much more complicated. In 2019 [16, 17] we managed to solve the problem of generalized localization not only for two-fold, but also for N-fold Fourier series. Let us give the exact formulation of this result:

Theorem 8.1 *Let $f \in L_2(\mathbb{T}^N)$ and $f = 0$ on some set $\Omega \subset \mathbb{T}^N$. Then the equality $\lim\limits_{\lambda\to\infty} S_\lambda f(x) = 0$ holds almost everywhere in Ω.*

It has been previously known (see [7]) that the generalized localization is not valid in $L_p(\mathbb{T}^N)$ when $1 \le p < 2$. Thus, the problem of generalized localization for the spherical partial sums is completely solved in $L_p(\mathbb{T}^N)$, $p \ge 1$: if $p \ge 2$ then we have the generalized localization and if $p < 2$, then the generalized localization fails.

As for the convergence almost everywhere of spherical partial sums for functions from the Sobolev classes, this issue for multiple Fourier integrals is also well studied by many specialists (see [6–15]). The most accurate result was obtained in the work of Carbery and Soria [14], where they proved that condition

$$a > (N-1)\left(\frac{1}{p} - \frac{1}{2}\right), \quad 1 < p \le 2 \tag{8.2}$$

ensures convergence almost everywhere for functions from the Sobolev classes $f \in L_p^a(\mathbb{R}^N)$. But the question of the validity of a similar result for multiple Fourier series remained unknown. This problem was recently solved in work [18] (see also the survey paper [19]). Let us formulate the corresponding result.

Theorem 8.2 *Let the parameters a and p satisfy conditions (8.2). Then, for any function $f \in L_p^a(\mathbb{T}^N)$, the partial sums $S_\lambda f(x)$ converge to $f(x)$ almost everywhere in $\mathbb{T}^N$. Moreover, the maximum operator $S_\star f(x) = \sup\limits_{\lambda>0} |S_\lambda f(x)|$ has the estimate*

$$||S_\star f||_{L_p(\mathbb{T}^N)} \le C_{p,a} ||f||_{L_p^a(\mathbb{T}^N)}. \tag{8.3}$$

It should be noted that the condition on the smoothness exponent a in this theorem coincides with the Stein condition (see [3]) on the Riesz mean exponent s.

In the next section, we will study the principle of generalized localization for spherical partial sums of multiple Fourier series, and Sect. 8.3 is devoted to the study of to the convergence almost everywhere of these same partial sums for smooth functions.

8.2 The Principle of Generalized Localization

By virtue of the definition of Fourier coefficients, we have

$$S_\lambda f(x) = \int\limits_{\mathbb{T}^N} \theta(x-y,\lambda) f(y)dy,$$

where the kernel can be represented in the form

$$\theta(x,\lambda) = (2\pi)^{-N} \sum_{|n|^2<\lambda} e^{inx}.$$

In order to study the convergence of $S_\lambda f(x)$ for $\lambda \to \infty$, it is necessary to study the asymptotic behavior of $\theta(x-y,\lambda)$ for large λ, which is far from simple task.

Obviously, when $\lambda \to \infty$ the kernel $\theta(x-y,\lambda)$ behaves best outside the diagonal $|x-y| > \delta > 0$ due to strong ossification. When studying the localization principle, we are precisely in this area, as a result of which the study of the localization principle of multiple Fourier series becomes a simpler task than the study of their convergence.

Let us now give a brief outline of the proof of Theorem 8.1. Let $f \in L_2(\mathbb{T}^N)$ and $f = 0$ in the ball $\{|x| < R\}$, $R < 1$. Then it is not difficult to verify that the theorem follows from the following estimate: for any $r < R$ there is a constant $C = C(R,r)$ such that

$$\int\limits_{|x|\le r} |S_\star f(x)|^2 dx \le C \int\limits_{\mathbb{T}^N} |f(x)|^2 dx. \tag{8.4}$$

Let $\chi_b(t)$ be the characteristic function of the segment $[0,b]$. Let $\varphi_1(t)$ denote a smooth function such that $\chi_{(R-r)/3}(t) \le \varphi_1(t) \le \chi_{2(R-r)/3}(t)$ and let us put $\varphi_2(t) = 1-\varphi_1(t)$. Define by $\psi(x)$ a function, 2π - periodic in each variable x_j, putting $\psi(x) = \varphi_2(|x|)$, $x \in \mathbb{T}^N$.

Let us denote $\theta_\lambda(x) = \theta(x,\lambda)\psi(x)$. Since the support of function f is the set $\{|x| \ge R\}$, then for any x from the ball $|x| \le r$ the equality

$$S_\lambda f(x) = \int\limits_{\mathbb{T}^N} \theta_\lambda(x-y) f(y)dy$$

holds. Therefore, if we denote this integral by $\theta_\lambda * f$, then to prove estimate (8.4) it is enough to establish the validity of the inequality

$$\int\limits_{\mathbb{T}^N} \sup_{j>0} \left|\theta_j * f\right|^2 dx \le C \int\limits_{\mathbb{T}^N} |f(x)|^2 dx, \tag{8.5}$$

where sup is taken over all natural numbers.

The Fourier coefficients $(\theta_j)_n$ of function $\theta_j(x)$ has the form

$$(\theta_j)_n = (2\pi)^{-2N} \int\limits_{\mathbb{T}^N} \sum_{|m|^2<j} e^{imx}\psi(x)e^{-inx}dx = (2\pi)^{-N} \sum_{|n-m|^2<j} \psi_m.$$

It is not hard to see that the following estimate

$$|(\theta_j)_n| \le \frac{C_l}{(1+||n|-\sqrt{j}|)^l} \tag{8.6}$$

is valid for any $l, j \in \mathbb{N}$ and $n \in \mathbb{Z}^N$ with some constant C_l depending on l, r and R.

Let us denote $(\Theta_j)_n = (\theta_{j+1})_n - (\theta_j)_n$, that is

$$(\Theta_j)_n = (2\pi)^{-N} \sum_{|m|^2=j} \psi_{m-n} = (2\pi)^{-N} \sum_{|n-m|^2=j} \psi_m.$$

Lemma 8.1 *There are sets Q_q^k, $q = 0, 1, \cdots, 2k-1$, of integers p, $0 \le p \le 2k$, such that for all l the estimate*

$$\sum_{q=0}^{2k-1} (q+1)^2 \sum_{p\in Q_q^k} |(\Theta_{k^2+p})_n|^2 \le \frac{C_l}{(1+\sqrt{||n|-k|})^l} \tag{8.7}$$

with some positive constant C_l is valid.

This lemma implies a uniform in $n \in \mathbb{Z}^N$ estimate:

$$\sum_{k=0}^{\infty} \sum_{q=0}^{2k-1} (q+1)^2 \sum_{p\in Q_q^k} |(\Theta_{k^2+p})_n|^2 \le C. \tag{8.8}$$

On the other hand, for the Fourier coefficients $(\theta_j)_n$, due to estimate (8.6), we have a uniform in $n \in \mathbb{Z}^N$ estimate

$$\sum_{k=0}^{\infty}\sum_{q=0}^{2k-1}(q+1)^{-2}\sum_{p\in Q_q^k}|(\theta_{k^2+p})_n|^2 \le C. \tag{8.9}$$

Now, we move on to the proof of the estimate (8.5). We have

$$[\theta_q * f]^2 = \sum_{j=0}^{q-1}[\Theta_j * f]^2 + 2\sum_{j=0}^{q-1}[\Theta_j * f][\theta_j * f].$$

Therefore,

$$\sup_{q\in\mathbb{N}}\left|\theta_q * f\right|^2 \le \sum_{j=0}^{\infty}|\Theta_j * f|^2$$

$$+\,2\sum_{k=0}^{\infty}\sum_{q=0}^{2k-1}\sum_{p\in Q_q^k}|\Theta_{k^2+p} * f|(q+1)|\theta_{k^2+p} * f|(q+1)^{-1}.$$

Integrating over $\mathbb{T}^N$, then using estimates (8.8) and (8.9), we have

$$\int_{\mathbb{T}^N}\sup_{q\in\mathbb{N}}\left|\theta_q * f\right|^2 \le \sum_n |f_n|^2\sum_{j=0}^{\infty}|(\Theta_j)_n|^2$$

$$+\sum_n |f_n|^2\sum_{k=0}^{\infty}\sum_{q=0}^{2k-1}(q+1)^2\sum_{p\in Q_q^k}|(\Theta_{k^2+p})_n|^2$$

$$+\sum_n |f_n|^2\sum_{k=0}^{\infty}\sum_{q=0}^{2k-1}(q+1)^{-2}\sum_{p\in Q_q^k}|(\theta_{k^2+p})_n|^2 \le C\sum_n |f_n|^2$$

$$= C\int_{\mathbb{T}^N}|f(x)|^2 dx.$$

Thus, estimate (8.5), and therefore Theorem 8.1 is proven.

8.3 Convergence of Fourier Series of Smooth Functions

In this section, we give a brief outline of the proof of Theorem 8.2.

Let us recall the definition of a Sobolev space. The function $f \in L_p(\mathbb{T}^N)$, $p \geq 1$, belongs to the Sobolev space $L_p^a(\mathbb{T}^N)$ with real number $a > 0$, if the norm

$$||f||_{L_p^a(T^N)} = \Big|\Big| \sum_{n \in Z^N} (1 + |n|^2)^{\frac{a}{2}} f_n e^{inx} \Big|\Big|_{L_p(T^N)} \tag{8.10}$$

is finite. When a is not an integer, then this space is also called a Liouville space. The norm in the Sobolev space $L_p^a(\mathbb{R}^N)$ is defined similarly to the norm (8.10):

$$||f||_{L_p^a(\mathbb{R}^N)} = \Big|\Big| \int_{\mathbb{R}^N} (1 + |\xi|^2)^{\frac{a}{2}} \hat{f}(\xi) e^{ix\xi} d\xi \Big|\Big|_{L_p(\mathbb{R}^N)}. \tag{8.11}$$

We also need a definition of the spherical partial Fourier integrals:

$$E_\lambda f(x) = (2\pi)^{-\frac{N}{2}} \int_{|\xi|^2 < \lambda} \hat{f}(\xi) e^{ix\xi} d\xi. \tag{8.12}$$

Here, the Fourier transform of the function $f \in L_2(\mathbb{R}^N)$ is defined by equality

$$\hat{f}(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} f(x) e^{-ix\xi} dx$$

and in this case, the integral by Plancherel's theorem converges in $L_2(\mathbb{R}^N)$.

Above, we noted the result of Carbery and Soria [14] on the convergence of spherical means of multiple Fourier integrals for functions from Sobolev classes with the same exponent a as in Theorem 8.2. A natural question arises: is it possible to obtain the statement of Theorem 8.2 by applying the well-known theorem of C.E. Kenig, P.A. Tomas [20] on the equiconvergence almost everywhere of the Fourier integrals and series? But the problem is that this theorem is only valid for functions from the classes $L_p(\mathbb{R}^N)$.

However, using the main idea of the work of C.E. Kenig and P.A. Tomas, it is possible to prove that the almost everywhere convergence of expansions E_λ for functions from Sobelev classes entails (although there is no equivalence as in [20]) the almost everywhere convergence of Fourier series S_λ.

In order to formulate the corresponding result, it is necessary first of all to move from spaces $L_p^a(\mathbb{T}^N)$ to classes $L_p(\mathbb{T}^N)$. To do this, using the Laplace operator Δ, we write

$$E_\lambda u = (1 - \Delta)^{-\frac{a}{2}} E_\lambda (1 - \Delta)^{\frac{a}{2}} u = (1 - \Delta)^{-\frac{a}{2}} E_\lambda v,$$

where $v = (1-\Delta)^{\frac{a}{2}} u \in L_p(\mathbb{R}^N)$. For any function $v \in L_p(\mathbb{R}^N)$, we introduce the operator

$$E_{(\lambda,a)} v = (1-\Delta)^{-\frac{a}{2}} E_\lambda v = (2\pi)^{-\frac{N}{2}} \int_{|\xi|^2<\lambda} (1+|\xi|^2)^{-\frac{a}{2}} \hat{v}(\xi) e^{ix\xi} d\xi.$$

Similarly, we define the operator $S_{(\lambda,a)} g$, $g \in L_p(\mathbb{T}^N)$ on the torus $\mathbb{T}^N$:

$$S_{(\lambda,a)} g = \sum_{|n|^2<\lambda} (1+|n|^2)^{-\frac{a}{2}} g_n e^{inx}, \quad g \in L_p(\mathbb{T}^N).$$

Let $E^\star_{(a)}$ and $S^\star_{(a)}$ be the corresponding maximal operators.

It is not difficult to verify that we cannot apply the Kenig and Tomas theorem to the pair of operators $E_{(\lambda,a)}$ and $S_{(\lambda,a)}$. Nevertheless, the following statement holds.

Theorem 8.3 *Let $1 < p < \infty$. If the operator $E^\star_{(a)}$ is bounded in $L_p(\mathbb{R}^N)$, then the operator $S^\star_{(a)}$ is bounded in $L_p(\mathbb{T}^N)$.*

Note that, in contrast to the Kenig and Tomas theorem, here only sufficient condition for the boundedness of the operators $S^\star_{(a)}$ is given. However, this is sufficient to prove the convergence of multiple Fourier series using the estimates of $E^\star_{(a)}$, obtained in [14].

Indeed, let $1 < p \le 2$, $q = \frac{2N}{N-1+2/p}$ and $a = \frac{N-1}{2}$. Application of Martsinkevich's interpolation theorem to the estimates obtained in paragraphs 3 and 4 of [14] gives

$$||E^\star_{(a)} u||_{L_q(\mathbb{R}^N)} \le C||u||_{L_p(\mathbb{R}^N)}, \quad u \in L_p(\mathbb{R}^N).$$

Consequently, $E^\star_{(a)} u$ is bounded almost everywhere in $\mathbb{R}^N$ for each function $u \in L_p(\mathbb{R}^N)$, $p > 1$. Then, by Stein's theorem on the sequence of linear operators, invariant under shift, acting from $L_p(\mathbb{R}^N)$ to $L_p(\mathbb{R}^N)$, $1 \le p \le 2$ (see [12] and [21]), it can be argued that

$$||E^\star_{(a)} u||_{L_{p,\infty}(\mathbb{R}^N)} \le C||u||_{L_p(\mathbb{R}^N)}, \quad f \in L_p(\mathbb{R}^N), \quad p > 1, \quad a = \frac{N-1}{2},$$

where $L_{p,\infty}(\mathbb{R}^N)$ are the Lorentz spaces.

Hence, by virtue of Theorem 8.3, for the same values of the parameters p and a, we have

$$||S^\star_{(a)} f||_{L_{p,\infty}(\mathbb{T}^N)} \le C||f||_{L_p(\mathbb{T}^N)}, \quad f \in L_p(\mathbb{T}^N). \tag{8.13}$$

On the other hand, from the estimates obtained in paragraph 3 of [14], it follows

$$||E^{\star}_{(a)}u||_{L_2(\mathbb{R}^N)} \leq C_a||u||_{L_2(\mathbb{R}^N)}, \quad u \in L_2(\mathbb{R}^N), \quad a > 0,$$

or, applying Theorem 8.3,

$$||S^{\star}_{(a)}f||_{L_2(\mathbb{T}^N)} \leq C_a||f||_{L_2(\mathbb{T}^N)}, \quad f \in L_2(\mathbb{T}^N), \quad a > 0. \tag{8.14}$$

Now, applying first the Martsinkevich interpolation theorem to estimates (8.13) and (8.14) (with $a = \frac{N-1}{2}$), we obtain

$$||S^{\star}_{(a)}f||_{L_p(\mathbb{T}^N)} \leq C||f||_{L_p(\mathbb{T}^N)}, \quad f \in L_p(\mathbb{T}^N), \quad p > 1, \quad a = \frac{N-1}{2}.$$

Then, applying to this estimate and estimate (8.14) (with $a > 0$) Stein's interpolation theorem on the family of linear operators analytically dependent on a parameter (see, for example, [3], p. 46) we will have

$$||S^{\star}_{(a)}f||_{L_p(\mathbb{T}^N)} \leq C_{p,a}||f||_{L_p(\mathbb{R}^N)}, \quad f \in L_p(\mathbb{T}^N),$$

where

$$1 < p \leq 2, \quad a > (N-1)(\frac{1}{p} - \frac{1}{2}).$$

Returning to Sobolev spaces, we rewrite this estimate in the form

$$||S_{\star}g||_{L_p(\mathbb{T}^N)} \leq C_{p,a}||g||_{L^a_p(\mathbb{T}^N)}, \quad g \in L^a_p(\mathbb{T}^N),$$

where p and a are the same as above.

This is estimate (8.3). The first part of Theorem 8.2 follows from this estimate.

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References

1. Carleson, L.: On convergence and growth of partial sums of Fourier series. Acta Math. **116**, 135–157 (1966)
2. Hunt, R.A.: On convergence of Fourier series. In: Proceedings of the Conference on Orthogonal Expansions and Their Continuous Analogues, pp. 235–255. University Press, Edwardsville–Carbondale (1968)
3. Alimov, S.A., Ashurov, R.R., Pulatov, A.K.: Multiple Fourier Series and Fourier Integrals. Commutat. Harmon. Anal. **VI**, 1–97 (1992)

4. Ilin, V.A.: On generalized interpretation of the localization principle for Fourier series in fundamental systems of functions. Siberian Math. J. **9**, 1093–1106 (1968)
5. Alimov, S.A., Ilin, V.A., Nikishin, E.M.: Convergence problems for multiple trigonometric series and spectral expansions. Progr. Math. Sci. **31**, 29–86 (1976)
6. Bastys, A.J.: The generalized localization principle for an N-fold Fourier integral. Sov. Math. Dokl. **278**, 777–778 (1984)
7. Bastys, A.J.: Generalized localization of Fourier series with respect to the eigenfunctions of the Laplace operator in the classes L_p. Litovskii Matematicheskii Sbornik **31**, 387–405 (1991)
8. Ashurov, R.R., Ahmedov, A., Mahmud Ahmad Rodzi, B.: The generalized localization for multiple Fourier integrals. J. Math. Anal. Appl. **371**, 832–841 (2010)
9. Ashurov, R.R., Butaev, A.: On generalized localization of Fourier inversion for distributions. Contemp. Math. **672**, 33–50 (2016)
10. Ashurov, R.R., Butaev, A., Pradhan, B.: On generalized localization of fourier inversion associated with an elliptic operator for distributions. Abstr. Appl. Anal. **2012**, 649848 (2012)
11. Carbery, A., Romera, E., Soria, F.: Radial weights and mixed norm inequalities for the disc multiplier. J. Funct. Anal. **109**, 52–75 (1992)
12. Carbery, A., Rubio de Francia, J.L., Vega, L.: Almost everywhere summability of Fourier integrals. J. Lond. Math. Soc. **38**, 513–524 (1988)
13. Carbery, A., Soria, F.: Almost everywhere convergence of Fourier integrals for functions in Sobolev spaces, and an L2-localization principle. Rev. Mat. Iberoam. **4**, 319–337 (1988)
14. Carbery, A., Soria, F.: Pointwise Fourier inversion and localization in R^n. J. Fourier Anal. Appl. **3**, 847–858 (1997)
15. Sjolin, P.: Regularity and integrability of spherical means. Monatsh. Math. **96**, 277–291 (1983)
16. Ashurov, R.R.: Generalized localization for spherical partial sums of multiple Fourier series. Rep. Russ. Acad. Sci. **489**, 7–10 (2019)
17. Ashurov, R.R.: Generalized localization for spherical partial sums of multiple Fourier series. J. Fourier Anal. Appl. **25**, 3174–3183 (2019)
18. Ashurov, R.R.: Convergence almost everywhere of multiple trigonometric Fourier series of functions from Sobolev classes. Math. Not. **109**, 163–169 (2021)
19. Ashurov, R.R.: Generalized localization and summability almost everywhere of multiple Fourier series and integrals. Mod. Math. Fundam. Direct. **67**, 634–653 (2021)
20. Kenig, C.E., Tomas, P.A.: Maximal operators defined by Fourier multipliers. Stud. Math. **68**, 79–83 (1980)
21. Stein, E.M.: Localization and summability of multiple Fourier series. Acta Math. **1–2**, 93–147 (1958)

Chapter 9
Regularization Methods for Solving Inverse Problems: A Comprehensive Review

Fadi Awawdeh

Abstract Inverse problems arise in various scientific and engineering disciplines, presenting challenges of ill-posedness and sensitivity to noise. This work conducts a comprehensive review of regularization methods aimed at stabilizing solutions to inverse problems. Focusing on techniques such as Tikhonov regularization, machine learning-based regularization, and Bayesian regularization, we explore their mathematical foundations, numerical implementations, and applications in diverse fields. Numerical implementation aspects, addressing discretization, stability, and computational efficiency, are presented to guide researchers in the practical application of regularization methods.

9.1 Introduction

The importance of regularization techniques in solving ill-posed inverse problems cannot be overstated. Regularization methods play a pivotal role in addressing the challenges associated with the sensitivity and instability of these problems. Given the tendency of ill-posed inverse problems to produce unreliable solutions due to noise or inaccuracies in measurements, regularization techniques act as a stabilizing force. By incorporating additional information or imposing constraints on the solution space, regularization helps prevent overfitting and guides the optimization process towards more realistic and meaningful solutions. Techniques such as Tikhonov regularization, total variation regularization, and sparsity-promoting methods provide a structured framework for balancing the fidelity to observed data and the stability of the solution [1]. Recently, powerful classes of regularization procedures have gained significant prominence and found noteworthy applications across various domains. These advanced regularization methods go beyond traditional techniques, offering more nuanced and effective ways to address

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the challenges associated with ill-posed inverse problems. Techniques such as machine learning-based regularization, Bayesian regularization, and deep learning regularization have emerged as cutting-edge approaches [2, 3].

In this work, we search the application of regularization techniques as a powerful methodology for addressing the challenges posed by ill-posed inverse problems. Additionally, the work explores the crucial aspect of choosing an appropriate regularization parameter, shedding light on methodologies for parameter selection that contribute to the method's adaptability and overall effectiveness. Through this investigation, we aim to provide insights that advance the understanding and application of regularization in the context of ill-posed inverse problems, contributing to the broader landscape of computational science and mathematical modeling.

9.2 Mathematical Framework

If one tackles an applied inverse problem by solving the corresponding operator equation $\mathcal{F}x = y$, formulated in infinite dimensional Banach spaces X and Y, the ill-posedness phenomenon appears. It produces serious practical problems since, instead of the exact right-hand side y, only noisy data, i.e., elements y^δ satisfying $\left\| y - y^\delta \right\| < \delta$ with noise level $\delta > 0$, are available. When using such perturbed data the goal consists of the stable approximate solution of the operator equation.

However, when the inverse operator $\mathcal{F}^{-1} : Y \to X$ lacks continuity, a critical issue arises. As the noise level $\delta \to 0$, y^δ converges to $\hat{y} = \mathcal{F}\big(\hat{x}\big)$, but the inverse reconstruction $\mathcal{F}^{-1}\left(y^\delta\right)$ does not necessarily converge to the true solution $\mathcal{F}^{-1}\left(\hat{y}\right) = \hat{x}$. Even the slightest amount of noise, as indicated by $\left\| y^\delta - \hat{y} \right\|_Y < \delta$, can significantly compromise the accuracy of the reconstruction process. This underscores the formidable challenge of achieving stable and reliable solutions in the presence of noise.

9.2.1 Regularization

In studying inverse problems

$$\mathcal{F} \colon X \to Y, \qquad \mathcal{F}(x) = y$$

with X, Y real Hilbert Spaces, the fundamental challenge lies in the lack of continuity of the inverse mapping $\mathcal{F}^{-1}$. For small perturbations y^δ, the conventional inverse $\mathcal{F}^{-1}(y^\delta)$ does not consistently converge to the true inverse $\mathcal{F}^{-1}(y)$ as $\delta \to 0$. To address this issue, the mathematical framework introduces a sequence of continuous approximations R_α to $\mathcal{F}^{-1}$, with the property that R_α converges to $\mathcal{F}^{-1}$ pointwise for $\alpha \to 0$.

Significantly, the parameter α is judiciously chosen as a function of δ, denoted as $\alpha(\delta)$. When this parameter choice is correctly made, the sequence $R_{\alpha(\delta)} y^{\delta}$ exhibits convergence to the true inverse $\mathcal{F}^{-1} y = x$ as $\delta \to 0$. This convergence is a critical aspect in applied inverse problems, where the interplay between inexact yet continuous reconstruction provided by R_{α} and the consideration of measurement noise through carefully chosen parameters is indispensable. This synergy proves pivotal for obtaining robust and accurate solutions, effectively navigating the challenges posed by ill-posedness and noisy data.

9.2.2 Smoothing

The optimization problem $\min_{x} \left\{ \left\| \mathcal{F}x - y^{\delta} \right\| \right\}$ is confronted with challenges arising from ill-posedness. This issue renders the problem potentially unsolvable or, if solutions exist, they can be unstable and inaccurate. Direct attempts to minimize the norm discrepancy are eschewed due to the tendency to produce highly oscillating solutions that lack physical realism. This observation has led to the development of a family of techniques that aim to enforce a certain degree of smoothness in the computed solution. This process goes by various names such as smoothing, regularization, or stabilization.

The fundamental idea behind these methods is to address the minimization problem associated with the equation $\mathcal{F}x = y^{\delta} = y + \delta$. Instead of directly minimizing the residual in the equation, expressed as $\left\| \mathcal{F}x - y^{\delta} \right\|$, a regularized minimization problem is formulated:

$$\min_{x} \left\{ \left\| \mathcal{F}x - y^{\delta} \right\| + \lambda \left\| x \right\|^{'} \right\}.$$

Here, $\left\| \cdot \right\|^{'}$, represents a potentially different norm from $\left\| \cdot \right\|$, and the positive scalar λ serves as a weighting factor. The simultaneous minimization of the residual and a norm of the solution introduces a damping effect that tends to suppress oscillations in the solution. The outcome is a smoothed solution that still satisfies the original equations to some extent.

The degree of smoothing is intricately linked to the choice of the norm $\left\| \cdot \right\|^{'}$ and the scalar λ. Regularization, in this context, can be seen as a delicate balance between ensuring that x^{δ} yields a small residual $\mathcal{F}x - y$ and constraining x^{δ} to be small in the chosen norm $\left\| \cdot \right\|^{'}$.

9.2.3 Auxiliary Problems

In the context of regularization, a crucial aspect involves the use of a regularization parameter $\lambda > 0$ to govern the properties of auxiliary problems. Larger values of λ signify higher stability in approximate solutions, albeit with a greater departure from the original problem. Conversely, values of λ approaching zero indicate auxiliary problems closely aligned with the original, but they tend to become increasingly unstable as λ approaches zero. Effective regularization approaches must navigate this delicate trade-off between conflicting goals—balancing stability and approximation.

9.3 Tikhonov Regularization

Tikhonov regularization begins with the assumption that the estimate of the noise level is known and can be expressed as $\left\|y - y^\delta\right\| \equiv \left\|\mathcal{F}x - y^\delta\right\| \leq \delta$, where x represents the exact solution, and y^δ is the known input data. The set $S = \left\{x : \left\|\mathcal{F}x - y^\delta\right\| \leq \delta\right\}$ encompasses the exact solution; however, due to the ill-posed nature of the problem, S is excessively large, encompassing both "good" solutions that approximate the exact one and undesirable, unphysical solutions.

To refine the selection of good solutions, Tikhonov regularization introduces the concept of selecting the smoothest function x from the set S. In more general terms, this is achieved through Tikhonov's stabilizing functional $\Omega(x)$. The minimization problem

$$\Omega(x) \to \min_x \text{ Subject to } \left\|\mathcal{F}x - y^\delta\right\| \leq \delta \tag{9.1}$$

is considered as a solution of $\mathcal{F}x = y$. Under sufficiently general conditions, the solution of Eq. (9.1) belongs to the boundary of the set S. Consequently, the original problem is reformulated as the minimization problem

$$\Omega(x) \to \min_x \text{ Subject to } \left\|\mathcal{F}x - y^\delta\right\| = \delta. \tag{9.2}$$

This reformulated problem (9.2) is solved using the method of Lagrange multipliers, leading to the minimization problem

$$M^\lambda(x) = \min_x \left\|\mathcal{F}x - y^\delta\right\|^2 + \lambda\Omega(x), \tag{9.3}$$

where the Lagrange multiplier is λ^{-1}. The parameter $\lambda > 0$ is determined from the condition $\left\|\mathcal{F}x^{\lambda} - y^{\delta}\right\| = \delta$, where x^{λ} is the solution of Eq. (9.3). The functional M^{λ} is referred to as Tikhonov's smoothing functional, and λ is termed the regularization parameter. The choice of Ω can vary depending on the available a priori information about the sought-for solution.

9.3.1 Choice of Regularization Parameter

Given a linear operator $\mathcal{F} : X \to Y$ and an observed data y^{δ} contaminated with noise, the discrepancy principle aims to find a regularization parameter α such that the norm of the difference between the observed data and the forward model $\mathcal{F}(x^{\alpha})$ is comparable to the expected noise level δ. Mathematically, this is expressed as $\left\|\mathcal{F}(x^{\alpha}) - y^{\delta}\right\| \approx \delta$, where $\|\cdot\|$ denotes a suitable norm in the space Y.

Consider a sequence of continuous approximations R_{α} to $\mathcal{F}^{-1}$, with the property that R_{α} converges to $\mathcal{F}^{-1}$ pointwise for $\alpha \to 0$. The goal is to ensure the convergence of the Tikhonov regularization, denoted as $R_{\alpha(\delta)}y^{\delta} \to \mathcal{F}^{-1}y$. The convergence condition is expressed as

$$\left\|R_{\alpha}y^{\delta} - \mathcal{F}^{-1}y\right\| \leq \underbrace{\left\|R_{\alpha}\left(y^{\delta} - y\right)\right\|}_{\|R_{\alpha}\|\delta} + \underbrace{\left\|R_{\alpha}y - \mathcal{F}^{-1}y\right\|}_{\to 0 \text{ for } \alpha \to 0}$$

To ensure convergence, we choose $\alpha(\delta)$ such that (for $\delta \to 0$) $\alpha(\delta) \to 0$ and $\left\|R_{\alpha(\delta)}\right\|\delta \to 0$.

An alternative and considered to be a better parameter choice is the discrepancy principle. The discrepancy principle involves choosing α such that $\left\|\mathcal{F}R_{\alpha}y^{\delta} - y^{\delta}\right\| \approx \delta$. This criterion ensures that the discrepancy between the forward model applied to the regularized solution and the observed data is approximately equal to the noise level.

9.3.2 An Application: Tikhonov Regularization Method for Determining the Heat Source

Consider the identification problem of heat source

$$\begin{cases} u_t - u_{xx} = f(x), & 0 < x < 1, \ 0 < t \leq 1, \\ u(x,0) = 0, & 0 \leq x \leq 1, \\ u_x(0,t) - u_x(1,t) = 0, & 0 \leq t \leq 1. \end{cases} \tag{9.4}$$

The problem here is that the source function $f(x)$ is unknown, which needs to be decided by some additional data. The additional data discussed here are observations at a final moment $t = 1$ given as follows:

$$u(x, 1) = g(x), \qquad 0 \leq x \leq 1. \tag{9.5}$$

By separation of variables, we obtain the solution of problem (9.4)–(9.5) as follows:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1 - e^{-n^2\pi^2 t}}{n^2\pi^2} \langle f, X_n\rangle X_n, \quad \langle f, X_n\rangle = \sqrt{2}\int_0^1 f(x)\cos(n\pi x)dx,$$

where $\left\{X_n = \sqrt{2}\cos n\pi x,\ n = 1, 2, \ldots,\right\}$ is an orthogonal basis in $L^2(0, 1)$. By the supplementary condition, we define the operator $K : f \to g$, then we have

$$g(x) = Kf(x) = \sum_{n=1}^{\infty} \frac{1 - e^{-n^2\pi^2}}{n^2\pi^2} \langle f, X_n\rangle X_n.$$

We can show that K is a linear, compact, and self-adjoint operator with

$$f(x) = K^{-1}g(x) = \sum_{n=1}^{\infty} \frac{n^2\pi^2}{1 - e^{-n^2\pi^2}} \langle g, X_n\rangle X_n.$$

In applications, the input data $g(x)$ can only be measured and never be exact. We assume the measured data function $g_\delta(x) \in L^2(0, 1)$ and satisfies $\|g - g_\delta\| \leq \delta$, where the constant $\delta > 0$ represents a noise level. We also impose an a priori bound on the heat source, i.e., $\|f\| \leq E$.

We will use Tikhonov regularization method to deal with the ill-posed problem (9.4) and (9.5), which can be rewritten as an operator equation $(Kf)(x) = g(x)$. Then, we give an approximate solution of $f(x)$ by a Tikhonov regularization method which minimizes the quantity $\|Kf - g_\delta\|^2 + \lambda^2 \|f\|^2$. To find the unique solution of this minimization problem, we solve the normal equation $K^*Kf_\delta(x) + \lambda^2 f_\delta(x) = K^*g_\delta(x)$, to obtain the explicit form

$$f_\delta(x) = \sum_{n=1}^{\infty} \frac{\frac{n^2\pi^2}{1-e^{-n^2\pi^2}}}{1 + \lambda^2\left(\frac{n^2\pi^2}{1-e^{-n^2\pi^2}}\right)^2} \langle g, X_n\rangle X_n.$$

We define a regularization approximate solution of problem (9.4) and (9.5):

$$f_{\lambda,\delta}(x) = \sum_{n=1}^{\infty} \frac{n^2\pi^2}{\left(1 - e^{-n^2\pi^2}\right)\left(1 + \lambda^2 n^4\right)} \langle g, X_n \rangle X_n.$$

Using the Discrepancy principle, we choose $\lambda = \left(\frac{\delta}{E}\right)^{1/(p+2)}$, $p > 0$. Then, the estimate $\left\| f - f_{\lambda,\delta} \right\| \leq C\delta^{p/(p+2)}$ holds.

9.4 Bayesian Regularization in Inverse Problems

Bayesian approaches, applied to regularization for solving inverse problems, offer a powerful probabilistic framework for seamlessly integrating prior knowledge with observed data. These methods, rooted in Bayesian principles, elegantly address uncertainty, providing stability and reliable solutions. Among its varied applications, Bayesian regularization finds extensive use in medical imaging, notably in the intricate domains of CT and MRI reconstruction. Additionally, it proves beneficial in signal processing tasks, where leveraging prior knowledge about signal characteristics is pivotal.

9.4.1 Bayesian Formulation

Bayesian regularization initiates by defining a prior distribution that encapsulates a priori information or beliefs concerning the solution. This distribution encapsulates knowledge before any data observation. Subsequently, the likelihood function is introduced, representing the probability of observing the given data given a specific solution. It succinctly encodes the information embedded in the data set.

9.4.2 Posterior Inference

Combining the prior distribution with the likelihood function, the Bayesian approach computes the posterior distribution. This updated distribution encapsulates refined beliefs about the solution after assimilating observed data. The posterior mean and variance furnish estimates of the solution, offering a quantifiable measure of associated uncertainty.

9.4.3 Types of Bayesian Regularization

- **Gaussian Processes (GP):** A prevalent tool in Bayesian regularization, GPs model solution spaces as distributions over functions, affording flexible and expressive priors.
- **Bayesian Neural Networks (BNN):** In the realm of deep learning, BNNs extend traditional networks by infusing uncertainty into weights, propagating this uncertainty through the entire network.

9.4.4 Bayesian Linear Regression Example

Let's consider a simple example of Bayesian regularization in linear regression. In this scenario, we have a data set of input-output pairs (x_i, y_i) and want to fit a linear model $y = mx + b$ to the data. However, we know from prior knowledge that the slope m should be close to a specific value m_0.

We assume a Gaussian prior on the slope m centered around m_0 with a certain variance, i.e.,

$$P(m) \propto \exp\left(-\frac{(m - m_0)^2}{2\sigma^2}\right).$$

This represents our prior belief that m is likely to be close to m_0.

Assuming Gaussian noise, the likelihood function is given by the product of normal distributions:

$$P\left(y|X, m, b\right) \propto \exp\left(-\frac{1}{2\sigma^2}\sum_i \left(y_i - mx_i - b\right)^2\right).$$

Note that the posterior distribution is proportional to the product of the prior and likelihood ($P\left(m|y, X, b\right) \propto P(m) \cdot P\left(y|X, m, b\right)$), the mean of the posterior distribution gives the Bayesian estimate for m, and the variance provides a measure of uncertainty. Here, the term $(m - m_0)^2$ in the prior acts as a regularization term penalizing deviations of m from m_0.

In this example, the prior effectively regularizes the estimate of the slope m, pulling it towards the prior mean m_0. The uncertainty in the estimate is captured by the width of the posterior distribution, see Fig. 9.1.

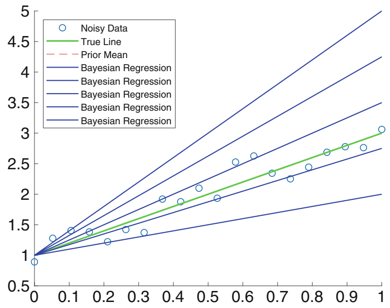


Fig. 9.1 Bayesian linear regression with regularization

9.5 Machine Learning Regularization

Machine learning models, while powerful, are susceptible to over fitting, a phenomenon where they memorize training data rather than discerning underlying patterns. To address this challenge, regularization techniques emerge as a crucial component, aiming to prevent over fitting and foster superior generalization to new, unseen data. The essence of regularization lies in striking a delicate balance between bias and variance. While a highly complex model may fit training data well (low bias), it risks failing to generalize (high variance). Regularization plays a pivotal role in finding an optimal trade-off between these competing factors.

9.5.1 Common Regularization Techniques

L1 Regularization, also known as Lasso (Least Absolute Shrinkage and Selection Operator), is a linear regression technique that introduces a regularization term to the linear regression objective function. The Lasso objective function is given by:

$$\text{Minimize} \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2 + \alpha \sum_{j=1}^{p} \left|\beta_j\right| .$$

Here, n is the number of observations, p is the number of features, y_i is the actual output, $\hat{y}_i$ is the predicted output, β_j is the coefficient of the j-th feature, and α is the regularization parameter, controlling the strength of the regularization.

Lasso has a feature selection property. It tends to drive some coefficients to zero, effectively performing automatic feature selection. This can be beneficial when dealing with high-dimensional data sets. Lasso regression is widely used in scenarios where feature selection is crucial, and a subset of features is expected to have a significant impact on the target variable.

L2 Regularization, also known as Ridge regularization, is a linear regression technique that prevents overfitting by penalizing large coefficients. It is called L2 regularization because it adds the squared magnitude of the coefficients to the objective function,

$$\text{Minimize} \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2 + \alpha \sum_{j=1}^{p} \beta_j^2.$$

Ridge regularization shrinks the coefficients towards zero but doesn't force them to be exactly zero. This is in contrast to Lasso regularization, which can lead to exact zero coefficients and feature selection. Ridge regularization is particularly useful when dealing with multicollinearity (high correlation between features). It helps stabilize the coefficients and improves the numerical stability of the model.

Elastic Net is a linear regression technique that combines both Lasso and Ridge regularization. It has the objective function:

$$\text{Minimize} \sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2 + \alpha \left(\lambda \sum_{j=1}^{p} \left|\beta_j\right| + \frac{1}{2} (1 - \lambda) \sum_{j=1}^{p} \beta_j^2 \right),$$

where α is the overall regularization strength and λ is the mixing parameter, controlling the balance between L1 and L2 regularization. The mixing parameter λ allows adjusting the balance between L1 and L2 regularization. Elastic Net retains the feature selection property of Lasso while benefiting from the stabilizing effect of Ridge.

9.5.2 Dropout in Neural Networks

Dropout is a regularization technique commonly used in neural networks to prevent overfitting. Dropout works as follows. During training, randomly selected neurons are "dropped out" or set to zero with a certain probability (typically between 0.2 and 0.5). The remaining neurons contribute to the forward pass, and the model learns with this reduced set of neurons. During testing, all neurons are used, but their weights are scaled by the dropout probability used during training. This scaling is applied to ensure that the expected output during testing remains similar to the expected output during training.

The dropout rate is a hyperparameter that determines the fraction of neurons to drop out during training. Dropout introduces noise during training, preventing the network from fitting the training data too closely and reducing overfitting. Dropout can be seen as training multiple sub-networks by randomly dropping out different sets of neurons. This has an ensemble effect, making the model more robust. The model becomes less sensitive to the specific weights of individual neurons, leading to better generalization to unseen data.

9.5.3 Additional Regularization Strategies

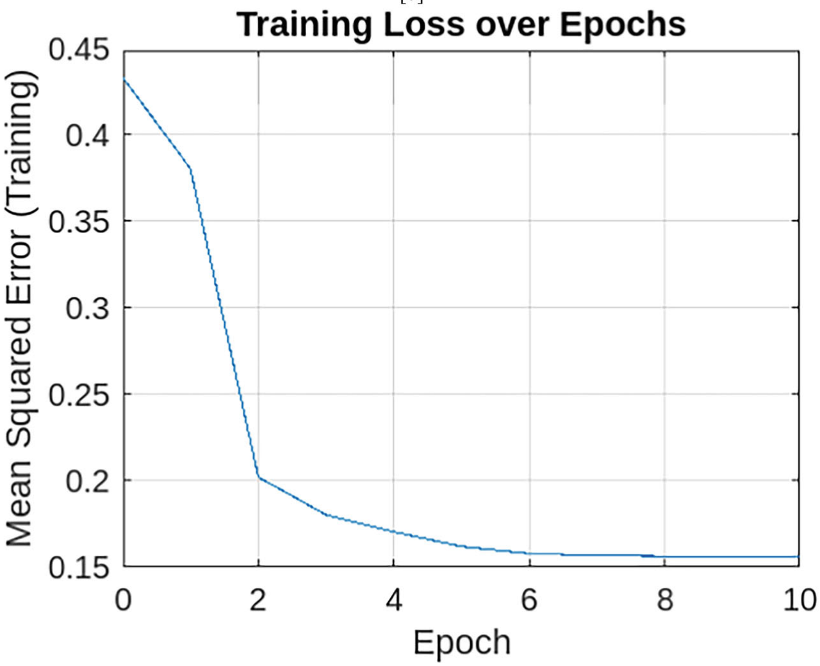
Early stopping is a strategic regularization approach that involves halting the training process once the model's performance on a validation set starts deteriorating. This preventative measure ensures that the model doesn't become overly specialized to the training data, maintaining its adaptability to diverse data sets.

Cross-validation is a crucial tool for evaluating how well a model will generalize to an independent data set. This process aids in tuning hyperparameters and selecting the optimal regularization strength, ensuring that the model performs optimally across various scenarios.

9.5.4 Neural Network Application

We present an example that demonstrates how to use early stopping to prevent overfitting and find an optimal stopping point during the training of a neural network. We create a neural network with one hidden layer containing 10 neurons. The data is split into training and validation sets. Early stopping is set up with parameters such as the maximum number of epochs, performance goal, and maximum number of validation failures before stopping. The neural network is trained using the training set, and early stopping is applied. The results are shown in Fig. 9.2.

[b]



[b]

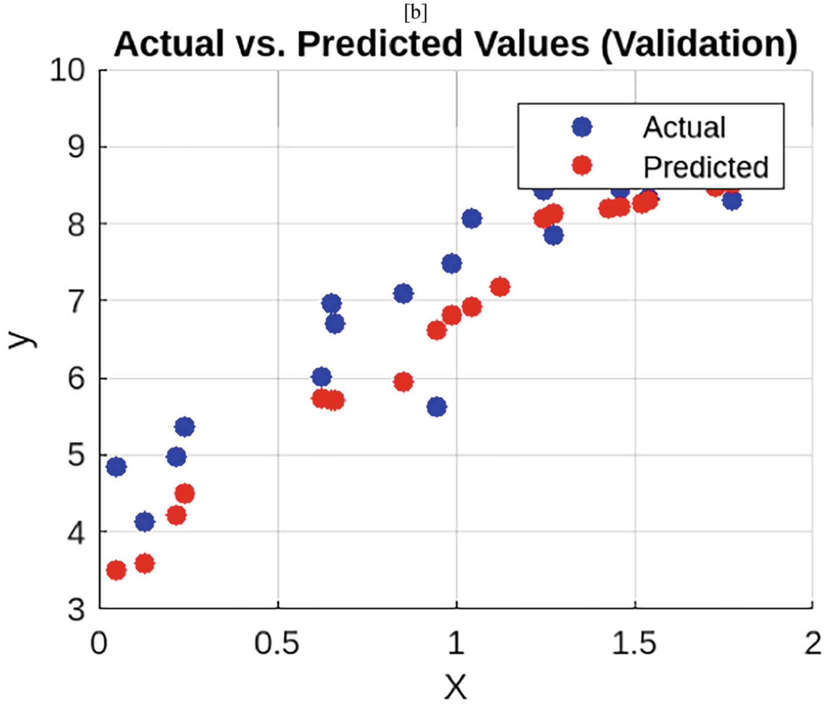


Fig. 9.2 Above: training loss over epochs, Below: actual vs. predicted values (validation)

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References

1. Kirsch, A.: An Introduction to the Mathematical Theory of Inverse Problems. Springer, New York (2011)
2. Tian, Y., Zhang, Y.: A comprehensive survey on regularization strategies in machine learning. Inf. Fusion **80**, 146–166 (2022)
3. Wüthrich, M.V., Merz, M.: Bayesian methods, regularization and expectation-maximization. In: Statistical Foundations of Actuarial Learning and its Applications. Springer Actuarial. Springer, Cham (2023)

Chapter 10
The Application of Spectral Resolution of a Self-Adjoint Operator to Approximate Elliptic Source Identification Problem with Neumann-Type Integral Condition

Charyyar Ashyralyyev and Aysel Cay

Abstract The paper focuses on the study of an elliptic source identification problem (SIP) with an integral condition for derivatives. Absolute stable difference scheme (DS) is proposed. Stability inequalities for solution of DS are established. Primarily DS is converted to auxiliary difference problem with some non local condition. Uniqueness and existence solution of DS is achieved by applying spectral resolution of a self-adjoint operator. Later, obtained results are used to establish stability inequalities for solution of DS for Neumann-type elliptic multidimensional SIP with integral condition. Finally, numerical illustration for 2D test example is carried out.

10.1 Introduction

It is well known that inverse problems are a class of mathematical problems where the goal is to determine the cause of a set of observations or measurements and they arise in various fields, including physics, engineering, medical imaging, geophysics, and many others. Theory and methods of solving such type of problems have been the focus of significant attention and thorough investigation by multiple researchers (see [1, 3–9, 15–17, 20–22, 24] and the references therein). The operator approach

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provides a rigorous mathematical foundation for stability analysis in a variety of contexts, contributing to the understanding of differential and difference problems in Banach and Hilbert spaces (see [2–11, 13–16] and the literature therein).

We consider a Hilbert space H, a self-adjoint positive definite operator (SAPDO) A on H. Let I denote the identity operator on H. Assume that specific function $f(\cdot) \in C^1([0,T],H)$, elements $\varphi, \psi, \zeta \in H$, number $\gamma \in [0,T]$ are given. The statement $A > \delta I$ for some positive number δ is related to the positive definiteness of A.

In paper [8], the authors established well-possedness of Neumann-type elliptic SIP with integral condition to obtain $p \in H$ and $u(\cdot) \in C^2([0,T],H) \cap C([0,T],D(A))$ so that

$$\begin{cases} -u_{tt}(t) + Au(t) = f(t) + p, \ t \in 0 < t < T, \\ u_t(T) = \int\limits_0^T \beta(s)\, u_s(s)ds + \psi, u_t(0) = \varphi, u(\gamma) = \zeta. \end{cases} \tag{10.1}$$

Introduce $[0,T]_\tau = \{t_i = i\tau, i = \overline{1,N}, \ \ N\tau = T\}$, a uniform grid space with sufficiently small $\tau > 0$. Denote $w_i = w(t_i), 0 \le i \le N$.

Let the given function β satisfy the following assumption

$$\sum_{i=1}^{N-1} |\beta_{i-1} - \beta_i| + |\beta_{N-1}| + |\beta_0| \le 1. \tag{10.2}$$

Since A is SAPDO, the operator $C = \frac{1}{2}(\tau A + \sqrt{4A + \tau^2 A^2})$ is also SAPDO [14, 18].

In [12] , the solution of difference scheme (DS)

$$-\tau^{-2}(v_{i+1} - 2v_i + v_{i-1}) + Av_i = f_i, \quad 1 \le i \le N-1$$

for given elements v_0 and v_N is represented by formula

$$\begin{aligned} v_i &= P\left[\left(\Delta_{i,2N-i}\right) v_0 + \Delta_{N-i,N+i} v_N\right] - P\Delta_{N-i,N+i} \\ &\times D\sum_{j=1}^{N-1} \Delta_{N-j,N+j} f_j \tau + D\sum_{j=1}^{N-1} \Delta_{|i-j|,i+j} f_j \tau\ , \ 1 \le i \le N-1, \end{aligned} \tag{10.3}$$

where

$$R = (I + \tau C)^{-1}, \ P = (I - R^{2N})^{-1}, \Delta_{i,j} = R^i - R^j,$$
$$D = (I + \tau C)\,(2I + \tau C)^{-1}\, C^{-1}.$$

Let $\alpha \in (0,1)$ be fixed number. We will use $C_\tau(H), C_\tau^\alpha(H)$, and $C_\tau^{\alpha,\alpha}(H)$, denote the Banach spaces of H-valued grid functions $w_\tau = \{w_k\}_{k=1}^{N-1}$ with the corresponding norms which described in [6], p.7 .

10.2 Absolute Stable DS for SIP

By using spectral resolution of a SAPDO A, it can be showed that, for every bounded continuous scalar function g, function of operator $g(A)$ is bounded and the estimate

$$\| g(A) \| \leq \sup_{\delta \leq \lambda < \infty} |g(\lambda)| \tag{10.4}$$

is valid [19, page 21]. Notice that here and in future for shortening we will use symbol $\| B \|$ instead of $\| B \|_{H \to H}$ for operator $B : H \to H$. Now, we give some lemmas which follow from spectral resolution of a SAPDO.

Lemma 10.1 ([12]) *The following estimates hold:*

$$\begin{aligned} &\| R^k \| \leq M(\delta)\,(1+\delta^{\frac{1}{2}}\tau)^{-k}, \quad \| CR^k \| \leq \tfrac{M(\delta)}{k\tau}, k \geq 1, \\ &\| P \| \leq M(\delta)\,, \ \delta > 0. \end{aligned} \tag{10.5}$$

Lemma 10.2 *The operator*

$$G = (I-R)^2 \left(R^{2N-2}\,(I+R)^2 - \left(I + R^{2N-1}\right)^2 \right)$$

is invertible, that is, there exists bounded G^{-1} so that

$$\| G^{-1} \| \leq M(\delta)\,. \tag{10.6}$$

Proof It is easy to see that

$$G = -\,(I-R)^2 \left(I - R^{2N-2}\right) P.$$

From (10.4) it implies that

$$\| (I-R)^{-2} \| \leq \sup_{\delta \leq \lambda < \infty} \left| \left[1 - \frac{1}{1 + \frac{\tau}{2}\left(\tau\lambda + \sqrt{4\lambda + \tau^2\lambda^2}\right)} \right]^{-2} \right| \leq M(\delta)\,,$$

$$\| \left(I - R^{2N-2}\right)^{-1} \| \leq \sup_{\delta \leq \lambda < \infty} \left| \left(1 - \frac{1}{\left[1 + \frac{\tau}{2}\left(\tau\lambda + \sqrt{4\lambda + \tau^2\lambda^2}\right)\right]^{2N-2}}\right)^{-1} \right|$$
$$\leq M(\delta). \tag{10.7}$$

Applying (10.5) and (10.7), we can establish that the operator G has inverse and estimate (10.6) is true. □

Lemma 10.3 *Suppose that assumption (10.2) is satisfied, then the following operator*

$$\begin{aligned} G_1 &= \Delta_{0,1}^2 \left(R^{2N-2}(I+R)^2 - \left(I + R^{2N-1}\right)^2\right) - \Delta_{0,1}\left(I + R^{2N-1}\right) \\ &\left\{- \sum_{i=1}^{N-2} \tau\left[\beta_{i-1} - \beta_i\right] \Delta_{N-i,N+i} + \left[\tau\left(\beta_{N-1} - \beta_{N-2}\right)\right] \Delta_{1,2N-1} \right. \\ &\left. -\tau\beta_{N-1}\Delta_{0,2N}\right\} - \Delta_{N-1,N+1}\left\{\tau\beta t_0 \Delta_{0,2N}\right. \\ &\left. +\tau\left(\beta_{N-1} - \beta_{N-2}\right)\Delta_{N-1,N+1} - \sum_{i=1}^{N-2} \tau\left[\beta_{i-1} - \beta_i\right]\Delta_{i,2N-i}\right\} \end{aligned} \tag{10.8}$$

has a bounded inverse G_1^{-1} such that

$$\| G_1^{-1} \|_{H \to H} \leq M(\delta). \tag{10.9}$$

Proof The operator G_1 can be rewritten as

$$\begin{aligned} &G_1 = G + \Delta_{0,1}\Delta_{0,2N} \sum_{i=1}^{N-2} \tau\left[\ \beta_{i-1} - \beta_i\right]\left(R^{N+i-1} \quad + R^{N-i}\ \right) \\ &-\tau\left(\beta_{N-1} - \beta_{N-2}\right)\left[\Delta_{0,1}\left(I + R^{2N-1}\right)\Delta_{1,2N-1} - \Delta_{N-1,N+1}^2\right] \\ &+\Delta_{0,1}\left(I + R^{2N-1}\right)\Delta_{0,2N}\tau\beta_{N-1} - \Delta_{0,2N}\Delta_{N-1,N+1}\tau\beta_0. \end{aligned}$$

Denote by $Q = G^{-1}$, $Q_1 = G_1^{-1}$. Then, the relation

$$Q_1 - Q = Q_1 Q S \tag{10.10}$$

is true, where

$$
\begin{aligned}
S &= \Delta_{0,1}\Delta_{0,2N}\sum_{i=1}^{N-2}\tau\left[\beta_{i-1}-\beta_i\right]\left(R^{N+i-1}+R^{N-i}\right)\\
&-\tau\left(\beta_{N-1}-\beta_{N-2}\right)\left[\Delta_{0,1}\left(I+R^{2N-1}\right)\Delta_{1,2N-1}-\Delta_{N-1,N+1}^2\right]\\
&+\Delta_{0,1}\left(I+R^{2N-1}\right)\Delta_{0,2N}\tau\beta_{N-1}-\Delta_{0,2N}\Delta_{N-1,N+1}\tau\beta_0.
\end{aligned}
$$

By using Cauchy Schwarz and triangle inequalities, we can get

$$
\begin{aligned}
&\|S\| \le \tau\left\|\Delta_{0,1}\right\|\left\|\Delta_{0,2N}\right\|\sum_{i=1}^{N-2}\left|\beta_{i-1}-\beta_i\right|\left\|R^{N+i-1}\right\|+\left\|R^{N-i}\right\|\\
&+\tau\left|\beta_{N-1}-\beta_{N-2}\right|\left[\left\|\Delta_{0,1}\right\|\left\|I+R^{2N-1}\right\|\left\|\Delta_{1,2N-1}\right\|+\left\|\Delta_{N-1,N+1}\right\|^2\right]\\
&+\tau\left\|\Delta_{0,1}\right\|\left\|I+R^{2N-1}\right\|_{H\to H}\left\|\Delta_{0,2N}\right\|\left|\beta_{N-1}\right|\\
&+\tau\left\|\Delta_{0,2N}\right\|\left\|\Delta_{N-1,N+1}\right\|\left|\beta_0\right|\\
&\le \tau M_1\sum_{i=1}^{N-1}\left|\beta_{i-1}-\beta_i\right|+M_2\left|\beta_{N-1}\right|+M_3\left|\beta_0\right|\le\tau M_4,
\end{aligned}
\tag{10.11}
$$

where, M_4 does not depend on τ. Thus, we can obtain

$$
\|Q_1\|\le\|Q\|+\|Q_1\|\,\|Q\|\,\|S\|\le M+\|Q_1\|\,MM_4\tau
$$

for any positive parameter τ. Thus, estimate (10.9) is proved. Now, we will associate primary SIP (10.1) with the appropriate first order of ADS

$$
\begin{cases}
-\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=f_k+p,\ 1\le k\le N-1;\\
u_1-u_0=\tau\varphi,\ u_l=\zeta,\ u_N-u_{N-1}=\sum\limits_{i=0}^{N-1}\tau\beta_i(u_{i+1}-u_i)+\tau\psi.
\end{cases}
\tag{10.12}
$$

Here, l is greatest integer part of $\frac{\gamma}{\tau}$. □

Theorem 10.1 *Suppose that assumption (10.2) is satisfied, $\varphi,\eta,\zeta\in D(A)$, and $f_\tau=\{f_k\}_{k=1}^{N-1}\in C_\tau^{\alpha,\alpha}(H)$ are given. Then, the solution (u_τ,p) of DS (10.12) obeys the following stability estimates*

$$
\|u_\tau\|_{C_\tau(H)}\le M(\delta)\left[\|\varphi\|_H+\|\zeta\|_H+\|\psi\|_H+\|f_\tau\|_{C_\tau(H)}\right],
\tag{10.13}
$$

$$
\left\|A^{-1}p\right\|_H\le M(\delta)\left[\|\varphi\|_H+\|\zeta\|_H+\|\psi\|_H+\|f_\tau\|_{C_\tau(H)}\right],
\tag{10.14}
$$

$$
\|p\|_H\le M(\delta)\left[\|A\varphi\|_H+\|A\zeta\|_H+\|A\psi\|_H+\frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(H)}\right].
\tag{10.15}
$$

Proof Applying

$$u_k = v_k + A^{-1} p, \tag{10.16}$$

one can obtain the auxiliary difference problem with two conditions as follows

$$\begin{cases} -\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2} + Av_k = f_k, \ 1 \le k \le N-1; \\ v_1 - v_0 = \tau\varphi, \quad \tau\beta_0 v_0 - \sum\limits_{i=1}^{N-2} \tau\left[\beta_{i-1} - \beta_i\right] \ v_i \\ + \left[-1 + \tau\left(\beta_{N-1} - \beta_{N-2}\right)\right] v_{N-1} + \left[1 - \tau\beta_{N-1}\right] v_N = \tau\psi. \end{cases} \tag{10.17}$$

First condition gives us relation

$$\left(\Delta_{1,2N-1} - \Delta_{0,2N}\right) v_0 + \Delta_{N-1,N+1} v_N = F_1 \tag{10.18}$$

with

$$\begin{aligned} F_1 = {} & \Delta_{N-1,N+1} D \sum_{j=1}^{N-1} \Delta_{N-j,N+j} f_j \tau \\ & - \Delta_{0,2N} D \sum_{j=1}^{N-1} \Delta_{|1-j|,1+j} f_j \tau + \tau \Delta_{0,2N} \varphi. \end{aligned}$$

Second condition gives us relation

$$\begin{aligned} & \Big\{ \tau\beta(t_0)(I - R^{2N}) - \sum_{i=1}^{N-2} \tau \left[\ \beta(t_{i-1}) - \beta(t_i)\right] \left(R^i - R^{2N-i}\right) \\ & + \left[-1 + \tau\left(\beta(t_{N-1}) - \beta(t_{N-2})\right)\right] \left(R^{N-1} - R^{N+1}\right) \Big\} v_0 \\ & + \Big\{ - \sum_{i=1}^{N-2} \tau \left[\ \beta(t_{i-1}) - \beta(t_i)\right] \left(R^{N-i} - R^{N+i}\right) + \left[1 - \tau\beta(t_{N-1})\right] \\ & \times (I - R^{2N}) + \left[-1 + \tau\left(\beta(t_{N-1}) - \beta(t_{N-2})\right)\right] \left(R - R^{2N-1}\right) \Big\} v_N = F_2 \end{aligned} \tag{10.19}$$

with

$$\begin{aligned} & F_2 = \tau \Delta_{0,2N} \psi + \sum_{i=1}^{N-2} \tau (\beta_{i-1} - \beta_i) \Bigg[\Delta_{N-i,N+i} D \sum_{j=1}^{N-1} \Delta_{N-j,N+j} f_j \tau \\ & - \Delta_{0,2N} D \sum_{j=1}^{N-1} \Delta_{|i-j|,i+j} f_j \tau \Bigg] + \left[-1 + \tau\left(\beta_{N-1} - \beta_{N-2}\right)\right] \\ & \times \left[\Delta_{1,2N-1} D \sum_{j=1}^{N-1} \Delta_{N-j,N+j} f_j \tau - \Delta_{0,2N} D \sum_{j=1}^{N-1} \Delta_{|N-1-j|,N-1+j} f_j \tau\right]. \end{aligned}$$

It is easy to get

$$
\begin{aligned}
&\Delta_{1,2N-1}-\Delta_{0,2N}=\Delta_{0,1}\left(I+R^{2N-1}\right),(\Delta_{1,2N-1}-\Delta_{0,2N})\\
&\quad\left[-\Delta_{1,2N-1}+\Delta_{0,2N})\right]\\
&+\Delta_{N-1,N+1}\Delta_{N-1,N+1}=\Delta_{0,1}^{2}\left(R^{2N-2}\left(I+R\right)^{2}-\left(I+R^{2N-1}\right)^{2}\right).
\end{aligned}
$$

It is invertible and has bounded inverse G_1^{-1}. So, solution of this linear system of equations can be derived by formula

$$
\begin{aligned}
v_0&=G_1^{-1}\left\{\left[\left[-\sum_{i=1}^{N-2}\tau\left[\beta i-1-\beta_i\right]\Delta_{N-i,N+i}+\left[-1+\tau\left(\beta_{N-1}-\beta_{N-2}\right)\right]\right.\right.\right.\\
&\times\Delta_{1,2N-1}+\left[1-\tau\beta_{N-1}\right]\Delta_{0,2N}\Big]\,\Delta_{N-1,N+1}D\sum_{j=1}^{N-1}\Delta_{N-j,N+j}f_j\tau\\
&-\Delta_{0,2N}D\sum_{j=1}^{N-1}\Delta_{|1-j|,1+j}f_j\tau\left[\Delta_{N-i,N+i}D\sum_{j=1}^{N-1}\Delta_{N-j,N+j}f_j\tau\right.\\
&+\tau\Delta_{0,2N}\varphi-\Delta_{N-1,N+1}\left[\tau\Delta_{0,2N}\psi+\sum_{i=1}^{N-2}\tau\left[\beta_{i-1}-\beta_i\right]\right.\\
&\times\left[\Delta_{1,2N-1}D\sum_{j=1}^{N-1}\Delta_{N-j,N+j}f_j\tau-\Delta_{0,2N}D\sum_{j=1}^{N-1}\Delta_{|i-j|,i+j}f_j\tau\right]\\
&\left.\left.\left.+\left[-1+\tau\left(\beta_{N-1}-\beta_{N-2}\right)\right]-\Delta_{0,2N}D\sum_{j=1}^{N-1}\Delta_{|N-1-j|,N-1+j}f_j\tau\right]\right]\right\}
\end{aligned}
\tag{10.20}
$$

and

$$
\begin{aligned}
v_N&=G_1^{-1}\left\{\left(R-R^{2N-1}-I+R^{2N}\right)\left[\tau(I-R^{2N})\psi+\sum_{i=1}^{N-2}\tau\left[\beta(t_{i-1})-\beta(t_i)\right]\right.\right.\\
&\times\left[\left(R^{N-i}-R^{N+i}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\right.\\
&\left.-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|i-j|}-R^{i+j}\right)f_j\tau\right]+\left[-1+\tau\left(\beta(t_{N-1})-\beta(t_{N-2})\right)\right]\\
&\times\left(R-R^{2N-1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau\\
&\left.-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|N-1-j|}-R^{N-1+j}\right)f_j\tau\right]-\tau\beta(t_0)(I-R^{2N})
\end{aligned}
$$

$$
-\sum_{i=1}^{N-2}\tau\left[\ \beta(t_{i-1})-\beta(t_i)\right]\left(R^{i}-R^{2N-i}\right)+\left[-1+\tau\left(\beta(t_{N-1})-\beta(t_{N-2})\right)\right]
$$

$$
\times\left(R^{N-1}-R^{N+1}\right)\left(R^{N-1}-R^{N+1}\right)D\sum_{j=1}^{N-1}\left(R^{N-j}-R^{N+j}\right)f_j\tau
$$

$$
\left.-(I-R^{2N})D\sum_{j=1}^{N-1}\left(R^{|1-j|}-R^{1+j}\right)f_j\tau+\tau(I-R^{2N})\varphi\right\}. \tag{10.21}
$$

Thus, it follows existence of a unique solution $\{v_k\}_{k=0}^{N}$ of difference problem (10.17). Solution is defined by (10.3), (10.20), and (10.21). By using formulas (10.3), (10.20),(10.21), estimates (10.5), (10.9), one can obtain

$$
\|v_\tau\|_{C_\tau(H)}\le M\left(\delta\right)\left[\|\varphi\|_H+\|\zeta\|_H+\|\psi\|_H+\|f_\tau\|_{C_\tau(H)}\right], \tag{10.22}
$$

$$
\begin{aligned}
&\left\|\{Av_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}+\left\|\left\{\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}\\
&\le M\left(\delta\right)\left[\frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(H)}+\|A\zeta\|_H+\|A\varphi\|_H+\|A\psi\|_H\right].
\end{aligned} \tag{10.23}
$$

The proofs of inequalities (10.14) and (10.15) for solution of difference problem (10.12) are based on formula (10.16) and corresponding estimates (10.22), (10.23). Finally, by using (10.14), (10.16), (10.22), we can achieve inequality (10.13). □

Theorem 10.2 *Assume that (10.2) is satisfied, $f_\tau\in C_\tau^{\alpha,\alpha}(H)$, and $\varphi,\zeta,\psi\in D(A)$. Then, for solution (u_τ,p) of difference problem (10.12) the coercive stability estimate*

$$
\begin{aligned}
&\left\|\left\{\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}+\left\|\{Au_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}+\|p\|_H\\
&\le M\left(\delta\right)\left[\frac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau^{\alpha,\alpha}(H)}+\|A\varphi\|_H+\|A\zeta\|_H+\|A\psi\|_H\right]
\end{aligned} \tag{10.24}
$$

holds.

The proof of inequality (10.24) is based on formulas (10.3), (10.16), (10.20), (10.21), and estimates (10.15), (10.23).

10.3 DS for Multidimensional Problem

In paper [8], the stability estimates for a solution of SIP for multidimensional elliptic equation

$$\begin{cases} -u_{tt}(t,x) - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}(t,x))_{x_r} + \sigma u(t,x) = f(t,x) + p(x), \\ (t,x) \in (0,T) \times \Omega, \\ u(\gamma,x) = \zeta(x), u_t(0,x) = \varphi(x), \\ u_t(T,x) = \int\limits_0^T \beta(s) u_s(s,x)\, ds + \psi(x), x \in \overline{\Omega}; \\ u(t,x) = 0, \ (t,x) \in [0,T] \times S \end{cases} \tag{10.25}$$

under all compatibility conditions were established. Here, $\Omega = (0,1)^n$ is the open unit cube in R^n with boundary S, $\overline{\Omega} = \Omega \cup S$; $a_r, \zeta, \varphi, \psi$, f are given smooth functions; $\forall x \in \Omega$, $a_r(x) \geq a_0 > 0$; $\sigma > 0, 0 < \gamma < T$ are known numbers. Abstract Theorems 10.1 and 10.2 allow us to get the appropriate first order of ADS for multidimensional SIP (10.25). Discretization of SIP (10.25) will be carried out in two steps. Firstly, for the grid spaces, we use the following notations:

$$\widetilde{\Omega}_h = \{x = (h_1 m_1, \ldots, h_n m_n); m = (m_1, \ldots, m_n),$$
$$m_i = \overline{0, M_i}, \ h_i M_i = 1, i = \overline{1,n} \},$$

$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S.$$

We define A_h^x as difference operator

$$A_h^x u^h(x) = -\sum_{i=1}^{n} \left(a_i(x) u_{\overline{x}_i}^h(x) \right)_{x_i, j_i} + \sigma u^h(x)$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$. It is known that the operator A_h^x is a SAPDO.

By using these notations, one can arrive at the next problem for a system of ordinary differential equations

$$\begin{array}{l} -\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x) + p^h(x), x \in \Omega_h, \ t \in (0,T), \\ \frac{du^h(0,x)}{dt} = \varphi(x); \ u^h(\gamma,x) = \zeta^h(x), \\ \frac{du^h(T,x)}{dt} - \int\limits_0^T \beta(\lambda) \frac{du^h}{ds}(s,x)\, ds = \psi^h(x), x \in \widetilde{\Omega}_h. \end{array} \tag{10.26}$$

Secondly, problem (10.26) is replaced by

$$\begin{cases} -\tau^{-2}\left[u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right] + A_h^x u_k^h(x) = f_k^h(x) + p^h(x), \\ 1 \leq k \leq N-1,\ x \in \Omega_h, \\ u_1^h(x) - u_0^h(x) = \tau\varphi^h(x), \\ u_N^h(x) - u_{N-1}^h(x) = \sum\limits_{i=0}^{N-1} \tau\beta_i\left(u_{i+1}^h(x) - u_i^h(x)\right) + \tau\psi^h(x), \\ \quad u_l^h(x) = \zeta^h(x)\ , x \in \widetilde{\Omega}_h. \end{cases} \tag{10.27}$$

For each $x \in \widetilde{\Omega}_h$, the value of $p^h(x)$ is defined by

$$p^h(x) = A_h^x\zeta^h(x) - A_h^x v^h(\gamma, x). \tag{10.28}$$

Denote by $L_{2h} = L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$, the Banach spaces of the grid functions $w^h(x) = \{w(h_1m_1, \cdots, h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the appropriate norms

$$\begin{aligned} &\left\|w^h\right\|_{L_{2h}} = (\textstyle\sum_{x\in\widetilde{\Omega}_h} |w^h(x)|^2 h_1\cdots h_n)^{1/2}, \\ &\left\|w^h\right\|_{W_{2h}^2} = \left\|w^h\right\|_{L_{2h}} + (\textstyle\sum_{x\in\widetilde{\Omega}_h}\sum_{i=1}^n \left|(w^h(x))_{x_i}\right|^2 h_1\cdots h_n)^{1/2} \\ &+(\textstyle\sum_{x\in\widetilde{\Omega}_h}\sum_{i=1}^n \left|(w^h(x))_{x_i\overline{x_i},m_i}\right|^2 h_1\cdots h_n)^{1/2}. \end{aligned}$$

Assume that both τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ are small positive fixed real numbers.

Theorem 10.3 *Suppose that inequality (10.2) is valid. Then, the solution u_τ of DS (10.27) exists and for solution, the stability inequalities hold:*

$$\begin{aligned} &\|u_\tau\|_{C_\tau(L_{2h})} \leq M(\delta)\,[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\zeta^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \|f_\tau\|_{C_\tau(L_{2h})}], \\ &\left\|p^h\right\|_{L_{2h}} \leq M(\delta)\left[\left\|\zeta^h\right\|_{W_{2h}^2} + \left\|\eta^h\right\|_{W_{2h}^2} + \left\|\phi^h\right\|_{W_{2h}^2} + \tfrac{1}{\alpha(1-\alpha)}\|f_\tau\|_{C_\tau(L_{2h})}\right]. \end{aligned}$$

Theorem 10.4 *Suppose that assumption (10.2) is satisfied, then for the solution of DS (10.27), the coercive stability inequality holds:*

$$\begin{aligned} &\left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^k)}{\tau^2}\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})} + \left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C_\tau(W_{2h}^2)} + \left\|p^h\right\|_{L_{2h}} \\ &\leq M(\delta)\,[\left\|\zeta^h\right\|_{W_{2h}^2} + \left\|\eta^h\right\|_{W_{2h}^2} + \left\|\phi^h\right\|_{W_{2h}^2} + \tfrac{1}{\alpha(1-\alpha)}\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}]. \end{aligned}$$

The proofs of these Theorems are based on the symmetry property of operator A_h^x in Hilbert space L_{2h} and the corresponding theorem in [23] on the coercive stability estimate for the solution of the elliptic difference problem in L_{2h} with first kind of boundary condition.

10.4 Numerical Illustration

Now, we will outlet numerical results for 2D test example of Neumann-type elliptic SIP with integral condition. Presented numerical results are carried out in framework of MATLAB.

It is easy to check that the pair of appropriate functions

$$(u(t,x), p(x)) = (\left(e^{-t} + t + 1\right) q(x), \left(\pi^2 + 1\right) q(x))(q(x) = \sin(\pi x))$$

is exact solution of the next 2D elliptic SIP:

$$\begin{cases} -u_{tt}(t,x) - u_{xx}(t,x) + u(t,x) = f(t,x) + p(x), \ 0 < t, x < 1, \\ u_t(0,x) = 0, \ u(0.2,x) = \zeta(x), u_t(1,x) = \int\limits_0^1 e^{-s} u_s(s) ds + \psi(x), \ 0 \le x \le 1, \\ u(t,0) = 0, \ u(t,1) = 0, \ 0 \le t \le 1. \end{cases} \tag{10.29}$$

Here, $f(t,x) = q(x)\left(-e^{-t} + \left(\pi^2+1\right)\left(e^{-t}+t\right)\right)$, $\psi(x) = \left[\frac{1}{2} - \frac{1}{2}e^{-2}\right] q(x)$, $\zeta(x) = \left(e^{-\frac{1}{5}} + \frac{6}{5}\right) q(x)$. The set of uniform grid points $[0,1]_\tau \times [0,1]_h$ is defined by

$$[0,1]_\tau \times [0,1]_h = \{(t_k, x_n) : t_k = k\tau, 0 \le k \le N, \ x_n = nh, 0 \le n \le M\},$$

which depends on sufficiently small parameters τ and h so that $N\tau = 1$, $Mh = 1$. Let $l = [0.2\tau]$, $\mu_0 = 0.2\tau - l$,

$$\varphi_n = 0, \psi_n = \psi(x_n), \zeta_n = \zeta(x_n); f_n^k = f(t_k, x_n), 0 \le k \le N, 0 \le n \le M.$$

Now, we will present algorithm with three stages for solving (10.29) approximately. In the 1st stage of algorithm, we search approximate solution of suitable auxiliary nonlocal boundary value problem (ANBVP). The 1st order of accuracy DS in t and the 2nd order of accuracy DS in x for appropriate ANBVP can be written as

$$\begin{cases} \frac{v_n^{k+1} - 2v_n^k + v_n^{k-1}}{\tau^2} + \frac{v_{n+1}^k - 2v_n^k + v_{n-1}^k}{h^2} - v_n^k = -f_n^k, 1 \le n \le M-1, 1 \le k \le N-1, \\ v_0^k = 0, v_M^k = 0, \ k = 0, \cdots, N, \ v_n^1 - v_n^0 = 0, \\ v_n^N - v_n^{N-1} = \sum\limits_{j=0}^{N-1} \tau e^{-t_i} \left(v_n^{j+1} - v_n^j\right) + \tau \psi_n, \ 0 \le n \le M. \end{cases} \tag{10.30}$$

Now, in the 2nd stage of algorithm, we calculate p_n by

$$p_n = -\frac{(\zeta_{n+1} - v_{n+1}^l) - 2(\zeta_n - v_n^l) + (\zeta_{n-1} - v_{n-1}^l)}{h^2} + (\zeta_n - v_n^l), \quad 1 \le n \le M-1.$$

DS (10.30) can be rewritten in the matrix form as follows

$$\begin{aligned} &Av_{n+1} + Bv_n + Cv_{n-1} = Ig^{(n)}, 1 \ \le n \le M-1, \\ &v_0 = \overrightarrow{0}, v_M = \overrightarrow{0}. \end{aligned} \tag{10.31}$$

Here, $g^{(n)}$ is a column matrix with $(N+1)$ elements, A, B, C are square matrices with $(N+1)^2$, and I is identity matrix, v_s is column matrix $v_s = \left[v_s^0 \ \ldots \ v_s^N \right]^t$, $s = n-1, n, n+1$. Denote by

$$a = \frac{1}{h^2}, c = \frac{1}{h^2}, \ q = -\frac{2}{h^2} - \frac{2}{\tau^2} - 1, r = \frac{1}{\tau^2},$$

$$A_n = C_n = diag(0, a, a, \ldots, a, 0).$$

Then,

$$\begin{aligned}
g_k^{(n)} &= -f(t_k, x_n), \ k = \overline{1, N-1}, \ n = \overline{1, M-1}, g_n^0 = \tau\varphi_n, \ g_n^N = \tau\psi_n, \\
&\quad n = \overline{1, M-1} \\
b_{i,i} &= q, b_{i-1,i} = r, b_{i,i-1} = r, i = \overline{2, N}; b_{11} = -1, b_{12} = 1, b_{N+1,N+1} \\
&= 1 - \frac{\tau e^{-t_{N-1}}}{2}, \\
b_{N+1,N} &= -1 - \frac{\tau e^{-t_{N-2}}}{2}, b_{N+1,1} = \left(\frac{3}{2} + \frac{e^{-t_1}}{2}\right)\tau, b_{N+1,2} = \left(-2 + \frac{e^{-t_2}}{2}\right)\tau, \\
b_{N+1,3} &= \left(\frac{1}{2} - \frac{e^{-t_1} - e^{-t_3}}{2}\right)\tau, \\
b_{N+1,j} &= -\frac{e^{-t_{j-1}} - e^{-t_{j+2}}}{2}\tau, \ j = 3, \ldots, N-1; b_{ij} = 0, \text{for other } i \text{ and } j.
\end{aligned}$$

Lastly, in the third stage, we define $\left\{u_n^k\right\}$ by $u_n^k = v_n^k + \zeta_n - v_n^l$. To find a numerical solution of (10.31), we apply a modification of the Gauss elimination method.

Calculated errors are presented in Tables 10.1, 10.2, and 10.3 for the first order of ADS in case of $(N, M) = (20, 20)$, $(N, M) = (40, 40)$, $(N, M) = (80, 80)$ and $(N, M) = (160, 160)$, respectively. Table 10.1 presents the error between the exact solution of NBVP and the solution derived by difference schemes.

Table 10.1 Error for auxiliary function v

DS/(N, M)	(20,20)	(40,40)	(80,80)	(160,160)
First order of ADS	7.09×10^{-3}	3.10×10^{-3}	1.44×10^{-3}	6.94×10^{-4}

Table 10.2 Error for function u

DS/(N, M)	(20,20)	(40,40)	(80,80)	(160,160)
First order of ADS	3.16×10^{-3}	1.50×10^{-3}	7.38×10^{-4}	3.65×10^{-4}

Table 10.3 Error for function p

Approximation/(N, M)	(20,20)	(40,40)	(80,80)	(160,160)
First order of ADS	3.23×10^{-2}	1.55×10^{-2}	7.60×10^{-3}	3.76×10^{-3}

Table 10.2 demonstrates values of error between exact and approximate solutions of u. Table 10.3 shows error for p.

10.5 Conclusion

In this paper, we study the approximation of an elliptic overdetermined second kind boundary value problem with integral condition. We construct the first order of ADS for the approximate solution of the overdetermined problem. By using spectral resolution of a self-adjoint operator, we establish stability inequalities for solutions of difference scheme. Later, these results are used to achieve stability inequalities for approximate solution of Neumann-type multidimensional source identification elliptic problem. Finally, we give numerical results for a 2D test example.

References

1. Akimova, E.N., Misilov, V.E., Sultanov, M.A.: Regularized gradient algorithms for solving the nonlinear gravimetry problem for the multilayered medium. Math. Methods Appl. Sci. **45**(15), 8760–8768 (2022)
2. Ashyraliyev, M., Ashyralyyeva, M.: Numerical solutions of source identification problem for hyperbolic-parabolic equation. AIP Conf. Proc. **1997**, 020048 (2018)
3. Ashyraliyev, M., Ashyralyyeva, M.: Note on the hyperbolic-parabolic identification problem with nonlocal conditions. AIP Conf. Proc. **2334**, 060001 (2021)
4. Ashyralyev, A.: A note on the Bitsadze-Samarskii type nonlocal boundary value problem in a Banach space. J. Math. Anal. Appl. **344**, 557–573 (2008)
5. Ashyralyyev, C.: Inverse Neumann problem for an equation of elliptic type. AIP Conf. Proc. **1611**, 46–52 (2014)
6. Ashyralyyev, C.: Numerical solution to Bitsadze-Samarskii type elliptic overdetermined multipoint NBVP. Bound. Value Probl. **2017**(74), 1–74 (2017)

7. Ashyralyyev, C., Akkan, Y.: Numerical solution to inverse elliptic problem with neumann type overdetermination and mixed boundary conditions. Electron. J. Differ. Equ. Conf. **201**(188), 1–15 (2015)
8. Ashyralyyev, C., Cay, A.: Well-posedness of Neumann-type elliptic overdetermined problem with integral condition. AIP Conf. Proc. **1997**, 020026 (2018)
9. Ashyralyyev, C., Dededurk, M.: Approximate solution of inverse problem for elliptic equation with overdetermination. Abstr. Appl. Anal. **2012**, 603018 (2012)
10. Ashyralyev, A., Ashyralyyev, C.: On the problem of determining the parameter of an elliptic equation in a Banach space. Nonlinear Anal. Model. Control **19**, 350–366 (2014)
11. Ashyralyev, A., Erdogan, A.S.: Parabolic time dependent source identification problem with involution and Neumann condition. Bull. Karaganda Univ. Math. **102**(2), 5–15 (2021)
12. Ashyralyev, A., Sobolevskii, P.E.: New Difference Schemes for Partial Differential Equations. Birkhäuser Verlag, Basel (2004)
13. Ashyralyev, A., Tetikoglu, F.S.O.: A note on Bitsadze-Samarskii type nonlocal boundary problems: well-posednesss. Numer. Funct. Anal. Optim. **34**, 939–975 (2013)
14. Ashyralyev, A., Tetikoglu, F.S.O.: On well-posedness of nonclassical problems for elliptic equations. Math. Methods Appl. Sci. **37**, 2663–2676 (2014)
15. Ashyralyev, A., Al-Hammouri, A., Ashyralyyev, C.: On the absolute stable difference scheme for the space-wise dependent source identification problem for elliptic-telegraph equation. Numer. Methods Partial Differ. Equ. **37**(2), 962–986 (2021)
16. Ashyralyev, A., Al-Hazaimeh, H., Ashyralyyev, C.: Absolute stability of a difference scheme for the multidimensional time-dependently identification telegraph problem. Comput. Appl. Math. **42**(8), 1–15 (2023)
17. Jenaliyev, M.T., Bektemesov, M.A., Yergaliyev, M.G.: On an inverse problem for a linearized system of Navier-Stokes equations with a final overdetermination condition. J. Inverse Ill-Posed Probl. **31**, (2023) https://doi.org/10.1515/jiip-2022-0065
18. Kabanikhin, S.I.: Inverse and Ill-Posed Problems: Theory and Applications. Walter de Gruyter, Berlin (2011)
19. Krein, S.G.: Linear Differential Equations in Banach Space. Nauka, Moscow (1966)
20. Orazov, I., Sadybekov, M.A.: On a class of problems of determining the temperature and density of heat sources given initial and final temperature. Sib. Math. J. **53**, 146–151 (2012)
21. Sadybekov, M., Oralsyn, G., Ismailov, M.: Determination of a time-dependent heat source under not strengthened regular boundary and integral overdetermination conditions. Filomat **32**, 809–814 (2018)
22. Skubachevskii, A.L.: On a nonlocal problem with integral boundary conditions for a multidimensional elliptic equation. Russ. Math. Surv. **71**, 801–906 (2016)
23. Sobolevskii, P.E.: Difference Methods for the Approximate Solution of Differential Equations. Voronezh State University Press, Voronezh (1975)
24. Zvonareva, T.A., Kabanikhin, S.I., Krivorotko, O.I.: Numerical algorithm for source determination in a diffusion–logistic model from integral data based on tensor optimization. Comput. Math. Math. Phys. **63**(9), 1654–1663 (2023)

Chapter 11
A Note on Numerical Solution of a Parabolic Source Identification Problem with Involution and Robin Condition

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Abstract This paper investigates a space source identification problem for parabolic equations involving involution and Robin conditions. The investigation establishes the well-posedness of the associated differential equation and introduces a stable difference scheme accompanied by stability estimates. Numerical results, serving as validation for the theoretical findings, are also presented.

11.1 Introduction

Several authors have explored numerical solutions and theoretical aspects of point source identification problems for parabolic equations, establishing well-posedness conditions and developing finite difference methods (see [1–9] and the references therein). Moreover, prior studies [10–14] investigated partial differential equations with involution, focusing on well-posedness and stability. This paper extends the results of [15, 16] to Robin boundary conditions. Robin boundary conditions represent a weighted combination of Dirichlet and Neumann boundary conditions. They are alternatively known as impedance boundary conditions, a term derived from their application in electromagnetic problems, or convective boundary conditions, owing to their relevance in heat transfer problems. If we think of heat conduction in a body, then the Neumann boundary condition describes an isolated body, while Robin conditions come into play when a portion of the heat is absorbed at the boundary [17].

Building upon the work of [15, 16], the paper contributes to study of a space source identification problem for parabolic equation with involution and Robin condition. The well-posedness theorem on the differential equation of the source

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identification parabolic problem is given. A stable difference scheme is constructed. Theoretical results are supported by a numerical experiment.

11.2 The Differential Problem and Its Stability

In this section, for the one dimensional parabolic differential equation we consider the space source identification problem with involution and Robin condition

$$\begin{cases} u_t(t,x) - (a(x)u_x(t,x))_x - \beta\left(a(-x)u_x(t,-x)\right)_x + \sigma u(t,x) \\ = p(x) + f(t,x), \quad -l < x < l, \ 0 < t < T, \\ u(t,-l) = \xi u_x(t,-l), \ 0 \le t \le T, \\ -u(t,l) = \mu u_x(t,l), \ 0 \le t \le T, \\ u(0,x) = \varphi(x), u(T,x) = \psi(x), \quad -l \le x \le l. \end{cases} \tag{11.1}$$

Problem (11.1) has a unique solution $(u(t,x), p(x))$ for the smooth functions $f(t,x)$ $((t,x) \in (0,T) \times (-l,l))$, $a \ge a(x) = a(-x) \ge \delta > 0$, $\delta - a|\beta| \ge 0$ $(x \in (-l,l))$, $\varphi(x)$, $\psi(x)$, $x \in [-l,l]$, $\xi > 0$, $\mu > 0$, and $\sigma > 0$.
Prior to giving the theorems on stability, we note that the Hilbert and Sobolev spaces used throughout the article were given in [16].

Theorem 11.1 *Suppose that* $\varphi, \psi \in W_2^2[-l,l]$. *Let* $f(t,x)$ *be continuously differentiable in* t *on* $[0,T] \times [-l,l]$ *function. Then, the solution of the identification problem (11.1) satisfies the stability estimates*

$$\|u\|_{C([0,T],L_2[-l,l])} + \left\|(A^x)^{-1} p\right\|_{L_2[-l,l]}$$

$$\le M_1(\delta,\sigma,\beta,l)\left[\|\varphi\|_{L_2[-l,l]} + \|\psi\|_{L_2[-l,l]} + \|f\|_{C([0,T],L_2[-l,l])}\right], \tag{11.2}$$

$$\|u\|_{C^{(1)}([0,T],L_2[-l,l])} + \|u\|_{C\left([0,T],W_2^2[-l,l]\right)} + \|p\|_{L_2[-l,l]}$$

$$\le M_2(\delta,\sigma,\beta,l)\left[\|\varphi\|_{W_2^2[-l,l]} + \|\psi\|_{W_2^2[-l,l]} + \|f\|_{C^{(1)}([0,T],L_2[0,l])}\right]. \tag{11.3}$$

Here, $M_1(\delta,\sigma,\beta,l)$ *and* $M_2(\delta,\sigma,\beta,l)$ *do not depend on* $\varphi(x)$, $\psi(x)$ *and* $f(t,x)$. *The Sobolev space* $W_2^2[-l,l]$ *is defined as the set of all functions* $u(x)$ *defined on* $[0,l]$ *such that* $u(x)$ *and the second order derivative function* $u''(x)$ *are all locally integrable in* $L_2[-l,l]$, *equipped the norm*

$$\|u\|_{W_2^2[-l,l]} = \left(\int_{-l}^{l} |u(x)|^2\, dx\right)^{\frac{1}{2}} + \left(\int_{-l}^{l} \left|u''(x)\right|^2 dx\right)^{\frac{1}{2}}.$$

Proof Problem (11.1) can be written in abstract form

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = p + f(t), 0 < t < T, \\ u(0) = \varphi, u(T) = \psi \end{cases} \tag{11.4}$$

in a Hilbert space $H = L_2[-l, l]$ with self-adjoint positive definite operator $A = A^x$ defined by the formula

$$A^x u(x) = -\left(a(x)u_x(x)\right)_x - \beta\left(a(-x)u_x\left(-x\right)\right)_x + \sigma u\left(x\right) \tag{11.5}$$

with the domain $D(A^x) = \left\{u \in W_2^2\left[-l, l\right] : u\left(-l\right) - \xi u\left(-l\right) = -u\left(l\right) - \mu u'\left(l\right) = 0\right\}$.
The proof of Theorem 11.1 is based on the symmetry properties of the space operator A as detailed in [10]. □

11.3 The Difference Scheme and Its Stability

In this section, for the approximate solution of identification problem (11.1), we consider a stable difference scheme beginning with the discretization of source identification problem (11.1) by defining the grid space

$$[-l, l]_h = \{x = x_n : x_n = nh,\ -M \le n \le M,\ Mh = l\}.$$

To the differential operator A of (11.5) generated for the differential problem, we assign the difference operator A_h^x by the formula

$$A_h^x \omega^h(x) = \{-(a(x)\omega_{\overline{x}}(x))_{x,r} - \beta\left(a(-x)\omega_{\overline{x}}(-x)\right)_{x,r} + \sigma\omega_r\}_{-M+1}^{M-1}, \tag{11.6}$$

acting in the space of grid functions $\omega^h(x) = \{\omega_r\}_{-M}^{M}$ satisfying the conditions $\omega_{-M} = -\xi\frac{\omega_{-M}-\omega_{-M+1}}{h}$ and $-\omega_M = \mu\frac{\omega_M-\omega_{M-1}}{h}$.
Using A_h^x, we reach to the following identification problem

$$\begin{cases} u_t^h(t, x) + A_h^x u^h(t, x) = p^h(x) + f^h(t, x),\ x \in [-l, l]_h\,,\ 0 < t < T, \\ u^h(0, x) = \varphi^h(x), u^h(T, x) = \psi^h(x),\ x \in [-l, l]_h\,. \end{cases} \tag{11.7}$$

Next, let's replace source identification problem (11.7) with the following first order of accuracy difference scheme

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau}+A_h^x u_k^h(x)=p^h(x)+f_k^h,\ f_k^h(x)=f(t_k,x), \\ t_k=k\tau,\ 1\le k\le N,\ N\tau=T,\ x\in[-l,l]_h\,, \\ u_0^h(x)=\varphi^h(x),\ u_N^h(x)=\psi^h(x),\ \ x\in[-l,l]_h\,. \end{cases} \tag{11.8}$$

We obtained the following stability estimate for the solution $\left\{\left\{u_k^h(x)\right\}_0^N,\ p^h(x)\right\}$ of problem (11.8).

Theorem 11.2 *The following stability estimates*

$$\left\|\left\{u_k^h\right\}_0^N\right\|_{C_\tau(L_{2h})}+\left\|\left(A_h^x\right)^{-1}p^h\right\|_{L_{2h}}$$

$$\le M_3(\delta,\sigma,\beta,l)\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}\right],$$

$$\left\|\left\{\frac{u_k^h-u_{k-1}^h}{\tau}\right\}_1^N\right\|_{C_\tau(L_{2h})}+\left\|\left\{u_k^h\right\}_0^N\right\|_{C_\tau(W_{2h}^2)}+\left\|p^h\right\|_{L_{2h}}\le M_4(\delta,\sigma,\beta,l)$$

$$\times\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N}\left\|\left\{\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\}_2^N\right\|_{L_{2h}}\right]$$

hold, where $M_3(\delta,\sigma,\beta,l)$ and $M_4(\delta,\sigma,\beta,l)$ do not depend on $\tau, h, f_k^h,\ 1\le k\le N,\ \varphi^h(x)$ and $\psi^h(x)$.

The proof of the theorem is based on the self- adjointness and positive definiteness of the space difference operator A in L_{2h} [10].

11.4 Numerical Results

Numerical methods serve as indispensable tools for achieving approximate results. In this section, we construct a first-order difference scheme for approximating the solution to a source identification problem, further supporting our theoretical findings with a numerical example.

We consider the identification problem with the Robin condition

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) - \frac{1}{2}u_{x,x}(t,-x) + u(t,x) \\ = p(x) - \sin x + \cos t \cos x + \frac{5}{2}\sin t \cos x + \sin t + \cos t, \\ \ x \in (-\pi,\pi), t \in (0,\pi), \\ u(0,x) = 0, u(\pi,x) = 0, x \in [-\pi,\pi], \\ u(t,-\pi) = u_x(t,-\pi), u(t,\pi) = -u_x(t,\pi), t \in [0,\pi] \end{cases} \tag{11.9}$$

for a parabolic equation with involution. The exact solution is

$$(u(t,x), p(x)) = (\sin t(1+\cos x), \sin x), -\pi \le x \le \pi, 0 \le t \le \pi.$$

For the numerical solution of source identification problem (11.9), we construct a difference scheme by the following steps. First, the set $[0,\pi]_\tau \times [-\pi,\pi]_h$ of all grid points is defined by

$$[0,\pi]_\tau \times [-\pi,\pi]_h = \{(t_k, x_n) : t_k = k\tau, 0 \le k \le N,$$

$$N\tau = \pi, x_n = nh, -M \le n \le M, Mh = \pi\}.$$

Second, we present the first order of accuracy difference scheme in t

$$\begin{cases} \tau^{-1}\left(u_n^k - u_n^{k-1}\right) - h^{-2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right) \\ -\frac{1}{2}h^{-2}\left(u_{-n+1}^k - 2u_{-n}^k + u_{-n-1}^k\right) + u_n^k = p_n - \sin x_n + \sin t_k + \cos t_k \\ + \cos t_k \cos x_n + \frac{5}{2}\sin t_k \cos x_n, 1 \le k \le N, -M+1 \le n \le M-1, \\ u_n^0 = 0, u_n^N = 0, -M \le n \le M, \\ u_{-M}^k + \frac{u_{-M}^k - u_{-M+1}^k}{h} = u_M^k + \frac{u_M^k - u_{M-1}^k}{h} = 0, 0 \le k \le N. \end{cases}$$

To get the approximate solution, in the first step, we obtain $\left\{\left\{w_n^k\right\}_0^N\right\}_{n=-M}^M$ as solution of nonlocal boundary value problem

$$\begin{cases} \tau^{-1}\left(w_n^k - w_n^{k-1}\right) - h^{-2}\left(w_{n+1}^k - 2w_n^k + w_{n-1}^k\right) \\ -\frac{1}{2}h^{-2}\left(w_{-n+1}^k - 2w_{-n}^k + w_{-n-1}^k\right) + w_n^k \\ = -\sin x_n + \cos t_k \cos x_n + \frac{5}{2}\sin t_k \cos x_n, 1 \le k \le N, -M+1 \le n \le M-1, \\ w_n^0 - w_n^N = 0, -M \le n \le M, \\ w_{-M}^k + \frac{w_{-M}^k - w_{-M+1}^k}{h} = w_M^k + \frac{w_M^k - w_{M-1}^k}{h} = 0, 0 \le k \le N, \end{cases} \tag{11.10}$$

where w_n^k denotes the numerical approximation of $w(t,x)$ at (t_k,x_n). The solution of difference scheme (11.10) is obtained after rewriting it in the following matrix form:

$$\left\{\begin{array}{l} Aw_{n+1}+Bw_n+Aw_{n-1}+Cw_{-n+1}+Dw_{-n}+Cw_{-n-1}=f_n, \\ Aw_{-n+1}+Bw_{-n}+Aw_{-n-1}+Cw_{n+1}+Dw_n+Cw_{n-1}=f_{-n}, \end{array}\right. , \tag{11.11}$$

$$1\le n\le M-1, \begin{pmatrix}(1+h)w_M\\(1+h)w_{-M}\end{pmatrix}=\begin{pmatrix}w_{M-1}\\w_{-M+1}\end{pmatrix}.$$

Here, w_s for $s=n,\ n\pm 1,$ and f_n are $(N+1)\times 1$ column matrices, and $(N+1)\times(N+1)$ square matrices A, B, C, D are defined as

$$A=\begin{bmatrix}0&0&0&.&0&0\\0&\kappa&0&.&0&0\\0&0&\kappa&.&0&0\\.&.&.&.&.&.\\0&0&0&.&\kappa&0\\0&0&0&.&0&\kappa\end{bmatrix},\quad B=\begin{bmatrix}1&0&0&.&0&0&-1\\\lambda&\mu&0&.&0&0&0\\0&\lambda&\mu&.&0&0&0\\.&.&\ddots&\ddots&.&.&.\\0&0&0&.&\lambda&\mu&0\\0&0&0&.&0&\lambda&\mu\end{bmatrix},$$

$$C=\frac{1}{2}A,\ D=-A.$$

Here, $\kappa=-\frac{1}{h^2},\ \lambda=-\frac{1}{\tau}$ and $\mu=\frac{1}{\tau}+\frac{2}{h^2}+1$. Grouping expression (11.11) as

$$\left\{\begin{array}{l} Aw_{n+1}+Cw_{-n-1}+Bw_n+Dw_{-n}+Aw_{n-1}+Cw_{-n+1}=f_n, \\ Cw_{n+1}+Aw_{-n-1}+Dw_n+Bw_{-n}+Cw_{n-1}+Aw_{-n+1}=f_{-n}, \end{array}\right. ,$$

and defining $Z_n=\begin{pmatrix}w_n\\w_{-n}\end{pmatrix}$ and $\phi_n=\begin{pmatrix}f_n\\f_{-n}\end{pmatrix}$, the system can be written as

$$\left\{\begin{array}{l} \begin{pmatrix}A&C\\C&A\end{pmatrix}Z_{n+1}+\begin{pmatrix}B&D\\D&B\end{pmatrix}Z_n+\begin{pmatrix}A&C\\C&A\end{pmatrix}Z_{n-1}=\phi_n,\ 1\le n\le M-1, \\ (1+h)Z_M=Z_{M-1}. \end{array}\right. \tag{11.12}$$

For solving system (11.12), we use the Gauss elimination method. To set up the Gauss elimination, we define

$$Z_n=\alpha_{n+1}Z_{n+1}+\beta_{n+1}, n=M-1,\dots,1, \tag{11.13}$$

where α_n $(1 \le n \le M)$ are $(2N+2)\times(2N+2)$ square matrices and β_n $(1 \le n \le M)$ are $(2N+2) \times 1$ column vectors, calculated as,

$$\begin{cases} \alpha_{n+1} = -\left(P\alpha_n + Q\right)^{-1} P, \\ \beta_{n+1} = \left(P\alpha_n + Q\right)^{-1} \left(I\phi_n - P\beta_n\right), \\ n = 1, \ldots, M-1, \end{cases} \tag{11.14}$$

where $P = \begin{pmatrix} A & C \\ C & A \end{pmatrix}$ and $Q = \begin{pmatrix} B & D \\ D & B \end{pmatrix}$ and I is $(2N+2) \times (2N+2)$ identity matrix.

Now, let's we evaluate α_n and β_n $(1 \le n \le M)$. Since,

$$\phi_0 = \begin{pmatrix} f_0 \\ f_0 \end{pmatrix} = \begin{pmatrix} Aw_1 + Cw_{-1} \\ Cw_1 + Aw_{-1} \end{pmatrix} + \begin{pmatrix} Bw_0 + Dw_0 \\ Dw_0 + Bw_0 \end{pmatrix} + \begin{pmatrix} Aw_{-1} + Cw_1 \\ Cw_{-1} + Aw_1 \end{pmatrix},$$

we get

$$Z_0 = \begin{pmatrix} w_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} B & D \\ D & B \end{pmatrix}^{-1} \left\{ - \begin{pmatrix} A+C & A+C \\ A+C & A+C \end{pmatrix} Z_1 + \phi_0 \right\}$$

and

$$\alpha_1 = -\begin{pmatrix} B & D \\ D & B \end{pmatrix}^{-1} \begin{pmatrix} A+C & A+C \\ A+C & A+C \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} B & D \\ D & B \end{pmatrix}^{-1} \phi_0.$$

We use iteration (11.14) to obtain α_n and β_n $(1 \le n \le M)$ values.

Since $(1+h)Z_M = Z_{M-1,}$ using formula (11.13), we first get

$$Z_M = (1+h)^{-1} Z_{M-1} = \left((1+h)^{-1} * I - \alpha_M\right)^{-1} \beta_M.$$

Next, using recursive formula (11.13), all $Z_n = \begin{pmatrix} w_n \\ w_{-n} \end{pmatrix}$ values are obtained.

We use the formula ([5, Equation 8]) to get p_n:

$$p_n = \frac{w_{n+1}^N - 2w_n^N + w_{n-1}^N}{h^2} + \frac{1}{2}\frac{w_{-n+1}^N - 2w_{-n}^N + w_{-n-1}^N}{h^2} - w_n^N$$

for $-M+1 \le n \le M-1$.

Table 11.1 Error analysis

Errors	$\|E_p\|_\infty$	$\|E_u\|_\infty$
$N = M = 20$	0.1852	0.0907
$N = M = 40$	0.0927	0.0396
$N = M = 80$	0.0464	0.0184
$N = M = 160$	0.0232	0.0089

Finally, by the formula (for details, see [5])

$$u_n^k = w_n^k - w_n^N, n = -M, -M+1, \ldots, M, k = 0, \ldots, N,$$

we obtain the numerical solution $\left\{ \{u_n^k\}_{k=0}^N \right\}_{n=-M}^M$.

Let $u(t,x)$ and $p(x)$ represent the exact solution, u_n^k represent the numerical solutions at (t_k, x_n), and p_n represent the numerical solutions at x_n. The maximum error between the exact solution and numerical solution is computed using the following formulas:

$$\begin{cases} \|E_u\|_\infty = \max\limits_{0 \le k \le N, -M \le n \le M} \left| u(t_k, x_n) - u_n^k \right|, \\ \|E_p\|_\infty = \max\limits_{-M < n < M} |p(x_n) - p_n|. \end{cases}$$

Error analysis for various N and M values are given in Table 11.1.

11.5 Conclusion

The present study investigated the source identification problem for a parabolic equation with involution and Robin boundary conditions. Employing mathematical tools, we established the well-posedness of the differential problem and derived stability estimates for a first-order difference scheme. To support our theoretical framework, a test problem was constructed and solved numerically using the modified Gauss elimination method. The resulting numerical values, summarized in the above table, exhibit the anticipated proportionality of errors to the step size, thereby providing strong validation for our analytical findings.

References

1. Choulli, M., Yamamoto, M.: Generic well-posedness of a linear inverse parabolic problem with respect to diffusion parameters. J. Inverse III-Pose. P. **7**(3), 241–254 (1999)
2. Ashyralyev, A., Agirseven, D.: On source identification problem for a delay parabolic equation. Nonlinear Anal. Model. **19**(3), 335–349 (2014)

3. Ashyralyev, A., Ashyralyyev, C.: On the problem of determining the parameter of an elliptic equation in a Banach space. Nonlinear Anal. Model. **19**(3), 350–366 (2014)
4. Ashyralyev, A., Erdogan A.S.: On the second order implicit difference schemes for a right hand side identification problem. Appl. Math. Comput. **226**, 212–228 (2014)
5. Ashyralyev, A., Erdogan, A.S., Demirdag, O.: On the determination of the right-hand side in a parabolic equation. Appl. Numer. Math. **62**(11), 1672–1683 (2012)
6. Ashyralyyev, C.: High order approximation of the inverse elliptic problem with Dirichlet-Neumann conditions. Filomat **28**(5), 947–962 (2014)
7. Blasio, G.D., Lorenzi, A.: Identification problems for parabolic delay differential equations with measurement on the boundary. J. Inverse Ill-Pose. P. **15**(7), 709–734 (2007)
8. Jator, S.: Block unification scheme for elliptic, telegraph, and Sine-Gordon partial differential equations. Am. J. Comput. Math. **5**(2), 175–185 (2015)
9. Ashyralyev, A., Sobolevskii, P.E.: New Difference Schemes for Partial Differential Equations. Birkhäuser Verlag, Basel (2004)
10. Ashyralyev, A., Sarsenbi, A.: Well-posedness of an elliptic equation with involution. Electron. J. Differ. Equ. **(2015)**, 1–8 (2015)
11. Ashyralyev, A., Sarsenbi, A.: Well-Posedness of a parabolic equation with involution. Numer. Funct. Anal. Opt. **38**(10), 1295–1304 (2017)
12. Ashyralyev, A., Sarsenbi, A.: Stability of a hyperbolic equation with the involution. In: Kalmenov, T.S., Nursultanov, E.D., Ruzhansky, M.V., Sadybekov M.A. (eds.) Functional Analysis in Interdisciplinary Applications. Springer Proceedings in Mathematics & Statistics, vol. 216, pp. 204–212. Springer, Berlin (2016)
13. Ashyralyev, A., Karabaeva, B., Sarsenbi, A.: Stable difference scheme for the solution of an elliptic equation with involution. AIP Conf. Proc. **1759**(1), 020111 (2016)
14. Cabada A., Tojo F.: Differential Equations with Involutions. Atlantis Press, Amsterdam (2015)
15. Ashyralyev, A., Erdogan, A.S., Sarsenbi, A.M.: A note on the parabolic identification problem with involution and Dirichlet condition. Bull. Karaganda Univ. Math. **99**(3), 130–139 (2020)
16. Ashyralyev, A., Erdogan, A.S.: Numerical solution of a parabolic source identification problem with involution and Neumann condition. In: Ashyralyev, A., Kalmenov, T.S., Ruzhansky, M.V., Sadybekov, M.A., Suragan, D. (eds.) Functional Analysis in Interdisciplinary Applications-II. ICAAM 2018. Springer Proceedings in Mathematics & Statistics, vol. 351. Springer, Cham (2021)
17. Povstenko, Y.Z.: Fundamental solutions to Robin boundary-value problems for the time-fractional heat-conduction equation in a half line. J. Math. Sci. **194**, 322–329 (2013)

Part III
Differential Equations and Their Applications

Chapter 12
On Sixth Order of Accuracy Four-Step Difference Schemes for the Fourth-Order Differential Equations

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Abstract Local and nonlocal boundary value problems for the fourth-order differential equations with dependent coefficients are studied. For solving these problems numerical solutions of novelly compact four-step difference schemes of sixth order of approximation generated by Taylor's decomposition on five points are presented. The theoretical statements for the solution of these difference schemes are supported by the results of numerical experiments.

12.1 Introduction

In applied sciences usually a highly accurate algorithm for the solution of problem is searched when exact solution could not be found. Hence, a task of current interest is the construction and investigation of highly accurate difference schemes for ordinary and partial differential equations with dependent coefficients. Application of Taylor's decomposition on two and three points for the numerical solution of compact finite difference schemes of high order approximation of linear ordinary and partial differential equations was well-investigated (see, for example [1–3]).

Taylor's decomposition on four and five points for the numerical solution a high-order of approximation compact finite difference schemes of linear ordinary and partial differential equations was not well-investigated. In papers [4, 5] and [6], three-step difference schemes of the fourth order of approximation generated by the Taylor's decomposition on four points for the numerical solutions of the local and nonlocal boundary-value problems for third-order ordinary and partial differential equations were investigated. Application of Taylor's decomposition on

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four points for the numerical solutions of third-order time-varying linear dynamical systems was presented. The method was illustrated for the numerical analysis of an up-converter used in communication systems. Taylor's decomposition on five points for the numerical solution of compact finite difference schemes of high order approximation of ordinary and partial differential equations was not well-investigated.

In paper [7], the four-step difference schemes generated by the Taylor's decomposition on five points of the sixth order of approximation for the numerical solutions of the boundary-value problem of the form

$$\begin{cases} \frac{d^4u(t)}{dt^4} + a(t)u(t) = f(t), 0 < t < T, \\ u(0) = \varphi, u'(0) = \psi, u(T) = \omega, u'(T) = \chi \end{cases} \tag{12.1}$$

for the fourth-order differential equation with dependent coefficient was presented. Here, and in future, $a(t)$ and $f(t)$ are given smooth functions defined on $[0, T]$.

In [8] , there has been an investigation of fourth and sixth order of accuracy difference schemes of numerical solutions for local problems of fourth-order ordinary differential equations, these problems were studied by using numerical solutions of high-order accurate compact finite difference schemes generated by Taylor decomposition on five points.

Boundary value problems for ordinary differential equations play a very important role in both theory and applications. They are used to describe a large number of physical, biological and chemical phenomena. The work of Timoshenko [9] on elasticity, the monograph by Soedel [10] on deformation of structures, and the work of Dulacska [11] on the effects of soil settlement are rich sources of such applications.

There has been a great deal of research work on boundary value problems for second and higher order differential equations, and we cite as recent contributions the papers of Anderson and Davis [12], Baxley and Haywood [13], Hao and Liu [14]. For surveys of known results and additional references we refer the reader to the monographs by Agarwal et al. [15, 16].

However, to the best of our knowledge, in the aforementioned papers and much other existing literature on ordinary differential equations mainly the multi-point boundary value problem for second order ordinary differential equations or the two-point boundary value problem for higher order ordinary differential equations were studied. There are very few works on the multi-point boundary value problem for higher order ordinary differential equations. For this reason, we are going to investigate the fourth order ordinary differential equation (12.1).

Nonlinear type version of Eq. (12.1), often referred to as the beam equation, has been studied under a variety of boundary conditions. A brief and easily accessible discussion and the physical interpretation for some of the boundary conditions associated with the linear beam equation can be found in the work of Zill and Cullen [17] . Multi-point boundary conditions of the type considered in this work are also somewhat different from the conjugate [18], focal [12, 19], and Lidstone [20] conditions that are commonly encountered in the literature.

Linear differential equations are subject to some boundary conditions arise in the mathematical description of some physical systems, for example, mathematical models of deflection of beams. These beams, which appear in many structures, deflect under their own weight or under the influence of some external forces. For example, if a load is applied to the beam in a vertical plane containing the axis of symmetry, the beam undergoes a distortion, and the curve connecting the centroids of all cross sections is called the deflection curve or elastic curve.

In the present paper, we study the four-step difference difference schemes of the high order of approximation for the numerical solutions of local boundary-value problem of the form

$$\begin{cases} \frac{d^4u(t)}{dt^4} + a(t)u(t) = f(t), 0 < t < T, \\ u(0) = \varphi, u'''(0) = \psi, u(T) = \omega, u'''(T) = \chi, \end{cases} \tag{12.2}$$

and nonlocal boundary-value problem of the form

$$\begin{cases} \frac{d^4u(t)}{dt^4} + a(t)u(t) = f(t), 0 < t < T, \\ u(0) = u(T) + \varphi, u'(0) = u'(T) + \psi, u''(0) = u''(T) + \omega, \\ u'''(0) = u'''(T) + \chi \end{cases} \tag{12.3}$$

for the fourth-order differential equation. We consider the five points $t_{k\pm 2}$, $t_{k\pm 1}$, t_k of the uniform grid space

$$[0, T]_\tau = \{t_k = k\tau, k = 0, 1, \cdots, N, N\tau = T\}.$$

The main aim of this work is the construction of high accurate four-step difference schemes for the numerical solution of fourth order differential equations. In the present paper, finite difference schemes of sixth order of approximation generated by Taylor's decomposition on five points for solving these problems are presented. The theoretical statements for the solution of these difference schemes are supported by the results of numerical experiments. The study of this paper is organized as follows. Section 12.1 is introduction. In Sects. 12.2 and 12.3, local and nonlocal boundary value problems (12.2), and (12.3) are considered. A novel numerical method for the solutions of these problems are investigated. Finally, Sect. 12.4 is conclusion and our future plans.

12.2 The Local Boundary Value Problem (12.2)

The construction of the sixth order of approximation for the approximate solution of problem (12.2) is based on the Taylor's decomposition on five points

Theorem 12.1 ([7]) *Let the functions $\nu(t)$ $(0 \leq t \leq T)$ have a tenth continuous derivative. Then the following relation holds:*

$$\tau^{-4}(\nu(t_{k+2}) - 4\nu(t_{k+1}) + 6\nu(t_k) - 4\nu(t_{k-1}) + \nu(t_{k-2})) \tag{12.4}$$

$$-\frac{237}{360}\nu^{(4)}(t_k) - \frac{31}{180}(\nu^{(4)}(t_{k+1}) + \nu^{(4)}(t_{k-1})) + \frac{1}{720}(\nu^{(4)}(t_{k+2}) + \nu^{(4)}(t_{k-2})) = o(\tau^6)$$

and sixth order of approximation formulas for $v'''(0)$ and $v'''(T)$.

Theorem 12.2 *Let the functions $\nu(t)(0 \leq t \leq T)$ have a thirteenth continuous derivative and $t_{k\pm2}$, $t_{k\pm1}$, $t_k \in [0, T]_\tau$. Then, the following relations hold*

$$v'''(0) = \tau^{-3}\left\{-\frac{801}{80}v(0) + \frac{349}{6}v(\tau) - \frac{18\,353}{120}v(2\tau) + \frac{2391}{10}v(3\tau)\right. \tag{12.5}$$

$$\left. -\frac{1457}{6}v(4\tau) + \frac{4891}{30}v(5\tau) - \frac{561}{8}v(6\tau) + \frac{527}{30}v(7\tau) - \frac{469}{240}v(8\tau)\right\} + o\left(\tau^6\right),$$

$$v'''(T) = \tau^{-3}\left\{\frac{801}{80}v(T) - \frac{349}{6}v(T-\tau) + \frac{18\,353}{120}v(T-2\tau)\right. \tag{12.6}$$

$$-\frac{2391}{10}v(T-3\tau) + \frac{1457}{6}v(T-4\tau) - \frac{4891}{30}v(T-5\tau) + \frac{561}{8}v(T-6\tau)$$

$$\left. -\frac{527}{30}v(T-7\tau) - \frac{469}{240}v(T-8\tau)\right\} + o\left(\tau^6\right).$$

Proof Applying the undetermined coefficients method, we will seek

$$v'''(0) = \alpha v(0) + \beta v(\tau) + \gamma v(2\tau) + dv(3\tau) + pv(4\tau) + qv(5\tau) \tag{12.7}$$

$$+ wv(6\tau) + mv(7\tau) + nv(8\tau) + o\left(\tau^6\right).$$

Using Taylor's formula, we get

$$v'(0) = \alpha v(0) + \beta \Big\{ v(0) + v'(0)\tau + v''(0)\frac{\tau^2}{2!} + v'''(0)\frac{\tau^3}{3!} + v^{(4)}(0)\frac{\tau^4}{4!}$$

$$+v^{(5)}(0)\frac{\tau^5}{5!} + v^{(6)}(0)\frac{\tau^6}{6!} + v^{(7)}(0)\frac{\tau^7}{7!} + v^{(8)}(0)\frac{\tau^8}{8!} + o\left(\tau^9\right)\Big\}$$

$$+\gamma \Big(v(0) + 2v'(0)\tau + 4v''(0)\frac{\tau^2}{2!} + 8v'''(0)\frac{\tau^3}{3!} + 16v^{(4)}(0)\frac{\tau^4}{4!}$$

$$+32v^{(5)}(0)\frac{\tau^5}{5!} + 64v^{(6)}(0)\frac{\tau^6}{6!} + (2)^7 v^{(7)}(0)\frac{\tau^7}{7!} + (2)^8 v^{(8)}(0)\frac{\tau^8}{8!} + o\left(\tau^9\right)\Big)$$

$$+d \Big(v(0) + 3v'(0)\tau + 9v''(0)\frac{\tau^2}{2!} + 27v'''(0)\frac{\tau^3}{3!} + 81v^{(4)}(0)\frac{\tau^4}{4!}$$

$$+243v^{(5)}(0)\frac{\tau^5}{5!} + 729v^{(6)}(0)\frac{\tau^6}{6!} + (3)^7 v^{(7)}(0)\frac{\tau^7}{7!} + (3)^8 v^{(8)}(0)\frac{\tau^8}{8!} + o\left(\tau^9\right)\Big)$$

$$+p \Big(v(0) + 4v'(0)\tau + (4)^2 v''(0)\frac{\tau^2}{2!} + (4)^3 v'''(0)\frac{\tau^3}{3!} + (4)^4 v^{(4)}(0)\frac{\tau^4}{4!}$$

$$+(4)^5 v^{(5)}(0)\frac{\tau^5}{5!} + (4)^6 v^{(6)}(0)\frac{\tau^6}{6!} + (4)^7 v^{(7)}(0)\frac{\tau^7}{7!} + (4)^8 v^{(8)}(0)\frac{\tau^8}{8!}$$

$$+o\left(\tau^9\right)\Big)$$

$$+q \Big(v(0) + 5v'(0)\tau + (5)^2 v''(0)\frac{\tau^2}{2!} + (5)^3 v'''(0)\frac{\tau^3}{3!} + (5)^4 v^{(4)}(0)\frac{\tau^4}{4!}$$

$$+(5)^5 v^{(5)}(0)\frac{\tau^5}{5!} + (5)^6 v^{(6)}(0)\frac{\tau^6}{6!} + (5)^7 v^{(7)}(0)\frac{\tau^7}{7!} + (5)^8 v^{(8)}(0)\frac{\tau^8}{8!}$$

$$+o\left(\tau^9\right)\Big)$$

$$+w \Big(v(0) + 6v'(0)\tau + (6)^2 v''(0)\frac{\tau^2}{2!} + (6)^3 v'''(0)\frac{\tau^3}{3!} + (6)^4 v^{(4)}(0)\frac{\tau^4}{4!}$$

$$+(6)^5 v^{(5)}(0)\frac{\tau^5}{5!} + (6)^6 v^{(6)}(0)\frac{\tau^6}{6!} + (6)^7 v^{(7)}(0)\frac{\tau^7}{7!} + (6)^8 v^{(8)}(0)\frac{\tau^8}{8!}$$

$$+o\left(\tau^9\right)\Big)$$

$$+ m \left(v(0) + 7v'(0)\tau + (7)^2 v''(0)\frac{\tau^2}{2!} + (7)^3 v'''(0)\frac{\tau^3}{3!} + (7)^4 v^{(4)}(0)\frac{\tau^4}{4!} \right.$$
$$+ (7)^5 v^{(5)}(0)\frac{\tau^5}{5!} + (7)^6 v^{(6)}(0)\frac{\tau^6}{6!} + (7)^7 v^{(7)}(0)\frac{\tau^7}{7!} + (7)^8 v^{(8)}(0)\frac{\tau^8}{8!}$$
$$\left. + o\left(\tau^9\right)\right)$$
$$+ n \left(v(0) + 8v'(0)\tau + (8)^2 v''(0)\frac{\tau^2}{2!} + (8)^3 v'''(0)\frac{\tau^3}{3!} + (8)^4 v^{(4)}(0)\frac{\tau^4}{4!} \right.$$
$$+ (8)^5 v^{(5)}(0)\frac{\tau^5}{5!} + (8)^6 v^{(6)}(0)\frac{\tau^6}{6!} + (8)^7 v^{(7)}(0)\frac{\tau^7}{7!} + (8)^8 v^{(8)}(0)\frac{\tau^8}{8!}$$
$$\left. + o\left(\tau^9\right)\right).$$

From that it follows

$$(-\alpha - \beta - \gamma - d - p - q - w - m - n)\, v(0)$$
$$+ (\beta + 2\gamma + 3d + 4p + 5q + 6w + 7m + 8n)\, v'(0)$$
$$+ \left(\frac{1}{2!}\beta + \frac{4}{2!}\gamma + \frac{9}{2!}d + \frac{(4)^2}{2!}p + \frac{(5)^2}{2!}q + \frac{(6)^2}{2!}w + \frac{(7)^2}{2!}m + \frac{(8)^2}{2!}n\right) v''(0)\,\tau^2$$
$$+ \left(\tau^{-3} - \left(\frac{1}{3!}\beta + \frac{8}{3!}\gamma + \frac{27}{3!}d + \frac{(4)^3}{3!}p + \frac{(5)^3}{3!}q + \frac{(6)^3}{3!}w + \frac{(7)^3}{3!}m + \frac{(8)^3}{3!}n\right)\right)$$
$$\times v'''(0)\,\tau^3$$
$$+ \left(\frac{1}{4!}\beta + \frac{16}{4!}\gamma + \frac{81}{4!}d + \frac{(4)^4}{4!}p + \frac{(5)^4}{4!}q + \frac{(6)^4}{4!}w + \frac{(7)^4}{4!}m + \frac{(8)^4}{4!}n\right) v^{(4)}(0)\,\tau^4$$
$$+ \left(\frac{1}{5!}\beta + \frac{32}{5!}\gamma + \frac{243}{5!}d + \frac{(4)^5}{5!}p + \frac{(5)^5}{5!}q + \frac{(6)^5}{5!}w + \frac{(7)^5}{5!}m + \frac{(8)^5}{5!}n\right)$$
$$\times v^{(5)}(0)\,\tau^5$$
$$+ \left(\frac{1}{6!}\beta + \frac{64}{6!}\gamma + \frac{729}{6!}d + \frac{(4)^6}{6!}p + \frac{(5)^6}{6!}q + \frac{(6)^6}{6!}w + \frac{(7)^6}{6!}m + \frac{(8)^6}{6!}n\right)$$
$$\times v^{(6)}(0)\,\tau^6$$

$$+\left(\frac{1}{7!}\beta+\frac{(2)^7}{7!}\gamma+\frac{(3)^7}{7!}d+\frac{(4)^7}{7!}p+\frac{(5)^7}{7!}q+\frac{(6)^7}{7!}w+\frac{(7)^7}{7!}m+\frac{(8)^7}{7!}n\right)$$
$$\times v^{(7)}(0)\,\tau^7$$
$$+\left(\frac{1}{8!}\beta+\frac{(2)^8}{8!}\gamma+\frac{(3)^8}{8!}d+\frac{(4)^8}{8!}p+\frac{(5)^8}{8!}q+\frac{(6)^8}{8!}w+\frac{(7)^8}{8!}m+\frac{(8)^8}{8!}n\right)$$
$$\times v^{(8)}(0)\,\tau^8+o\left(\tau^9\right).$$

Then equating the coefficients τ^8 , $0 \le p \le 8$ of the lowest power of τ to zero in the last formula, we obtain the system of equations

$$\begin{cases}
-\alpha-\beta-\gamma-d-p-q-w-m-n=0,\\
\beta+2\gamma+3d+4p+5q+6w+7m+8n=0,\\
\frac{1}{2!}\beta+\frac{(2)^2}{2!}\gamma+\frac{(3)^2}{2!}d+\frac{(4)^2}{2!}p+\frac{(5)^2}{2!}q+\frac{(6)^2}{2!}w+\frac{(7)^2}{2!}m+\frac{(8)^2}{2!}n=0,\\
\frac{1}{3!}\beta+\frac{(2)^3}{3!}\gamma+\frac{(3)^3}{3!}d+\frac{(4)^3}{3!}p+\frac{(5)^3}{3!}q+\frac{(6)^3}{3!}w+\frac{(7)^3}{3!}m+\frac{(8)^3}{3!}n=\tau^{-3},\\
\frac{1}{4!}\beta+\frac{16}{4!}\gamma+\frac{81}{4!}d+\frac{(4)^4}{4!}p+\frac{(5)^4}{4!}q+\frac{(6)^4}{4!}w+\frac{(7)^4}{4!}m+\frac{(8)^4}{4!}n=0,\\
\frac{1}{5!}\beta+\frac{32}{5!}\gamma+\frac{243}{5!}d+\frac{(4)^5}{5!}p+\frac{(5)^5}{5!}q+\frac{(6)^5}{5!}w+\frac{(7)^5}{5!}m+\frac{(8)^5}{5!}n=0,\\
\frac{1}{6!}\beta+\frac{64}{6!}\gamma+\frac{729}{6!}d+\frac{(4)^6}{6!}p+\frac{(5)^6}{6!}q+\frac{(6)^6}{6!}w+\frac{(7)^6}{6!}m+\frac{(8)^6}{6!}n=0,\\
\frac{1}{7!}\beta+\frac{(2)^7}{7!}\gamma+\frac{(3)^7}{7!}d+\frac{(4)^7}{7!}p+\frac{(5)^7}{7!}q+\frac{(6)^7}{7!}w+\frac{(7)^7}{7!}m+\frac{(8)^7}{7!}n=0,\\
\frac{1}{8!}\beta+\frac{(2)^8}{8!}\gamma+\frac{(3)^8}{8!}d+\frac{(4)^8}{8!}p+\frac{(5)^8}{8!}q+\frac{(6)^8}{8!}w+\frac{(7)^8}{8!}m+\frac{(8)^8}{8!}n=0.
\end{cases}$$

Solving this system of equations, we obtain $\alpha=-\frac{801}{80\tau^3}$, $\beta=\frac{349}{6\tau^3}$, $\gamma=-\frac{18\,353}{120\tau^3}$, $d=\frac{2391}{10\tau^3}$, $p=-\frac{1457}{6\tau^3}$, $q=\frac{4891}{30\tau^3}$, $w=-\frac{561}{8\tau^3}$, $m=\frac{527}{30\tau^3}$, $n=-\frac{469}{240\tau^3}$. So, relation (12.5) is proved. In the similar manner, we can prove the relation (12.6). The proof of Theorem 12.2 is finished.

Applying Taylor's decomposition on five points to Eqs. (12.1) and (12.2), formulas (12.5), (12.6) and neglecting small terms, we get the sixth order approximation four-step difference scheme

$$\begin{cases} \frac{1}{\tau^4}\left(u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}\right)+\frac{237}{360}a(t_k)u_k \\ +\frac{31}{180}\left(a(t_{k+1})u_{k+1}+a(t_{k-1})u_{k-1}\right)-\frac{1}{720}\left(a(t_{k+2})u_{k+2}+a(t_{k-2})u_{k-2}\right) \\ =\frac{237}{360}f(t_k)+\frac{31}{180}\Big(f(t_{k+1})+f(t_{k-1})-\frac{1}{720}\left(f(t_{k+2})+f(t_{k-2})\right)\Big), \\ 2\le k\le N-2,\ u_0=\varphi,\ u_N=\omega, \\ \tau^{-3}(-\frac{801}{80}u_0+\frac{349}{6}u_1-\frac{18\,353}{120}u_2+\frac{2391}{10}u_3-\frac{1457}{6}u_4+\frac{4891}{30}u_5 \\ -\frac{561}{8}u_6+\frac{527}{30}u_7-\frac{469}{240}u_8)=y'''(0), \\ \tau^{-3}(\frac{801}{80}u_N-\frac{349}{6}u_{N-1}+\frac{18\,353}{120}u_{N-2}-\frac{2391}{10}u_{N-3}+\frac{1457}{6}u_{N-4}-\frac{4891}{30}u_{N-5} \\ +\frac{561}{8}u_{N-6}-\frac{527}{30}u_{N-7}+\frac{469}{240}u_{N-8})=y'''(T) \end{cases} \tag{12.8}$$

for numerical solution of the boundary-value problem (12.2). For numerical analysis, we consider boundary-value problem (12.2) for the simple case when $T = 1, a\,(t) = 1, \varphi = \psi = \omega = \chi = 0$, and

$$f(t) = 4! - 4\cdot 5!t + 6\cdot\frac{6!}{2!}t^2 - 4\cdot\frac{7!}{3!}t^3 + \frac{8!}{4!}t^4 + t^4\,(1-t)^4\,.$$

Then, the exact solution is

$$u(t) = t^4\,(1-t)^4\,.$$

Noted that for the approximate solution of this problem (12.2), we use the sixth order of accuracy difference scheme (12.8) with different values of τ. Here and in future, the error of the numerical solutions is defined by formula (12.9).

$$E_N = \max_{0\le k\le N} |u(t_k) - u_k|. \tag{12.9}$$

Noted that if N is doubled, the values of errors between the exact solution and approximate solution decreases by a factor of approximately $1/64$ for difference scheme (12.8). So, the order of approximation of this difference scheme is 6 (see, Table 12.1).

Table 12.1 Errors of difference scheme (12.8) with different values of τ

$\tau = \frac{1}{N}$	$N = 10240$	$N = 20480$
Difference Scheme (12.8)	0.4368	0.0146

12.3 The Nonlocal Boundary Value Problem (12.3)

We consider nonlocal boundary-value problem (12.3). The construction of the sixth order of approximation for the approximate solution of problem (12.3) is based on Theorems 12.1 and 12.2 and on the sixth order of approximation formulas for $v'(0)$, $v'(T)$, $v''(0)$ and $v''(T)$.

Theorem 12.3 ([9]) *Let the functions $v(t)(0 \leq t \leq T)$ have a 7-th continuous derivative. Then, the following relations hold*

$$v'(0) - \tau^{-1}\left\{-\frac{49}{20}v(0) + 6v(\tau) - \frac{15}{2}v(2\tau) + \frac{20}{3}v(3\tau)\right. \tag{12.10}$$

$$\left. -\frac{15}{4}v(4\tau) + \frac{6}{5}v(5\tau) - \frac{1}{6}v(6\tau)\right\} = o(\tau^6),$$

$$v'(T) - \tau^{-1}\left\{\frac{49}{20}v(T) - 6v(T-\tau) + \frac{15}{2}v(T-2\tau) - \frac{20}{3}v(T-3\tau)\right. \tag{12.11}$$

Theorem 12.4 ([8]) *Let the functions $v(t)$ have a 8-th continuous derivative. Then, the following relations hold:*

$$v''(0) - \tau^{-2}\left\{\frac{469}{90}v\,(0) - \frac{223}{10}v\,(\tau) + \frac{879}{20}v\,(2\tau) - \frac{949}{18}v\,(3\tau)\right. \tag{12.12}$$

$$\left. +41v\,(4\tau) - \frac{201}{10}v\,(5\tau) + \frac{1019}{180}v\,(6\tau) - \frac{7}{10}v\,(7\tau)\right\} = o(\tau^6),$$

$$v''(T) - \tau^{-2}\left\{\frac{469}{90}v\,(T) - \frac{223}{10}v\,(T-\tau) + \frac{879}{20}v\,(T-2\tau) - \frac{949}{18}v\,(T-3\tau)\right. \tag{12.13}$$

$$\left. +41v\,(T-4\tau) - \frac{201}{10}v\,(T-5\tau) + \frac{1019}{180}v\,(T-6\tau) - \frac{7}{10}v\,(T-7\tau)\right\} = o(\tau^6).$$

$$\left. +\frac{15}{4}v(T-4\tau) - \frac{6}{5}v(T-5\tau) + \frac{1}{6}v(T-6\tau)\right\} = o(\tau^6).$$

Applying Taylor's decomposition on five points to Eqs. (12.2) and (12.3), Theorems 12.3 and 12.4 and formulas (12.10), (12.11), (12.5) and (12.6) and neglecting *small terms, we get the sixth order of approximation difference scheme*

$$\begin{cases} \frac{1}{\tau^4}\left(u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}\right)+\frac{237}{360}a(t_k)u_k \\ +\frac{31}{180}\left(a(t_{k+1})u_{k+1}+a(t_{k-1})u_{k-1}\right)-\frac{1}{720}\left(a(t_{k+2})u_{k+2}+a(t_{k-2})u_{k-2}\right) \\ =\frac{237}{360}f(t_k)+\frac{31}{180}\left(f(t_{k+1})+f(t_{k-1})-\frac{1}{720}\left(f(t_{k+2})+f(t_{k-2})\right)\right), \\ 2\le k\le N-2, u_0=u_N+\varphi, \\ \tau^{-1}(-\frac{49}{20}u_0+6u_1-\frac{15}{2}u_2+\frac{20}{3}u_3-\frac{15}{4}u_4+\frac{6}{5}u_5-\frac{1}{6}u_6)-\psi \\ =\tau^{-1}(\frac{49}{20}u_N-6u_{N-1}+\frac{15}{2}u_{N-2}-\frac{20}{3}u_{N-3}+\frac{15}{4}u_{N-4}-\frac{6}{5}u_{N-5}+\frac{1}{6}u_{N-6}), \\ \tau^{-2}(\frac{469}{90}u_0-\frac{223}{10}u_1+\frac{879}{20}u_2-\frac{949}{18}u_3+41u_4-\frac{201}{10}u_5+\frac{1019}{180}u_6-\frac{7}{10}u_7) \\ =\tau^{-2}(\frac{469}{90}u_N-\frac{223}{10}u_{N-1}+\frac{879}{20}u_{N-2}-\frac{949}{18}u_{N-3}+41u_{N-4}-\frac{201}{10}u_{N-5} \\ +\frac{1019}{180}u_{N-6}-\frac{7}{10}u_{N-7})+\omega, \\ \tau^{-3}(-\frac{801}{80}u_0+\frac{349}{6}u_1-\frac{18\,353}{120}u_2+\frac{2391}{10}u_3-\frac{1457}{6}u_4+\frac{4891}{30}u_5-\frac{561}{8}u_6 \\ +\frac{527}{30}u_7-\frac{469}{240}u_8)=\tau^{-3}(\frac{801}{80}u_N-\frac{349}{6}u_{N-1}+\frac{18\,353}{120}u_{N-2}-\frac{2391}{10}u_{N-3} \\ +\frac{1457}{6}u_{N-4}-\frac{4891}{30}u_{N-5}+\frac{561}{8}u_{N-6}-\frac{527}{30}u_{N-7}+\frac{469}{240}u_{N-8})+\chi \end{cases} \tag{12.14}$$

for the numerical solution of nonlocal boundary-value problem (12.3). For numerical analysis, we consider the boundary-value problem

$$\begin{cases} \frac{d^4u(t)}{dt^4}+u(t)=4!-4\cdot 5!t+6\cdot\frac{6!}{2!}t^2-4\cdot\frac{7!}{3!}t^3 \\ +\frac{8!}{4!}t^4+t^4\left(1-t\right)^4, 0<t<1, \\ u(0)-u(1)=u'(0)-u'(1)=u''(0)-u''(1)=u'''(0)-u'''(1)=0 \end{cases} \tag{12.15}$$

with the exact solution

$$u(t)=t^4\left(1-t\right)^4.$$

Noted that for the approximate solution of this problem , we used the fourth order of accuracy difference schemes (12.14) with different values of τ.

Table 12.2 Errors of difference scheme (12.14) with different values of τ

$\tau = \frac{1}{N}$	$N = 20$	$N = 40$
Difference Scheme (12.14)	0.0139	4.8026e-05

Noted that if N is doubled, the values of errors between the exact solution and approximate solution decreases by a factor of approximately $1/64$ *for difference scheme (12.14). So, the order of approximation of this difference scheme is 6 (see, Table 12.2).*

12.4 Conclusion and Our Future Plans

1. In this article, we study local and nonlocal boundary value problems for the fourth-order differential equations with dependent coefficients. Finite difference schemes of sixth order of approximation generated by Taylor's decomposition on five points for solving these problems are constructed and investigated. Numerical results are supported by the results of numerical experiments.
2. Construct and investigate high accurate four-step difference schemes for the numerical solution of the local and nonlocal problems for the fourth order general differential equations
$$\frac{d^4u(t)}{dt^4} + d(t)\frac{d^3u(t)}{dt^3} + c(t)\frac{d^2u(t)}{dt^2} + b(t)\frac{du(t)}{dt} + a(t)u(t) = f(t), 0 < t < T.$$
Here, $a(t), b(t), c(t), d(t),$ and $f(t)$ be given smooth functions defined on $[0, T]$.
3. Construct and investigate high accurate four-step difference schemes for the numerical solution of the local and nonlocal problems for the abstract fourth order elliptic differential equations
$$\frac{d^4y(t)}{dt^4} + A\,(t)\,y(t) = f(t), 0 < t < T$$
in a Hilbert space H, with the self adjoint positive definite operators $A\,(t)$. Note that operator method of [1] will permit us to establish the stability of these difference schemes.

References

1. Ashyralyev, A., Sobolevskii, P.E.: New Difference Schemes for Partial Differential Equations. Birkhäuser Verlag, Basel (2004)
2. Ashyralyev, A., Sobolevskii, P.E.: On the two new approaches for construction of the high order of approximation difference schemes for the second order differential equations. Function. Differ. Equ. **10**(3–4), 333–405 (2003)
3. Ashyralyev, A., Sobolevskii, P.E.: On the two-step the high order of approximation difference schemes for the second order differential equations. Proc. Dynam. Syst. Appl. **4**, 528–535 (2004)
4. Ashyralyev, A., Arjmand, D.: A note on the Taylor's decomposition on four points for a third order differential equation. Appl. Math. Comput. **188**(2), 1483–1490 (2007)
5. Ashyralyev, A., Arjmand, D., Koksal, M.: Taylor's decomposition on four points for solving third-order linear time-varying systems. J. Franklin Inst. Eng. Appl. Math. **346**(7), 651–662 (2009)
6. Arjmand, D.: Highly accurate difference schemes for the numerical solution of third-order ordinary and partial differential equations. MS Thesis, KTH Royal Institute of Technology (2010)
7. Ashyralyeva, M.A.: A note on the Taylor's decomposition on five points and its applications to differential equations. Funct. Differ. Equ. **13**(3–4), 357–370 (2006)
8. Ashyralyev, A., Ibrahım, I.M.: High order of accurate finite difference schemes for fourth-order differential equations. Axioms **13**(1), 34 (2024)
9. Timoshenko, S.P.: Theory of Elastic Stability. McGraw-Hill, New York (1961)
10. Soedel, W.: Vibrations of Shells and Plates. Dekker, New York (1993)
11. Dulácska, E.: Soil settlement effects on buildings. In: Developments in Geotechnical Engineering, vol. 69. Elsevier, Amsterdam (1992)
12. Anderson, D.R., Davis, J.M.: Multiple solutions and eigenvalues for third-order right focal boundary value problem. J. Math. Anal. Appl. **267**, 135–157 (2002)
13. Baxley, J., Haywood, L.J.: Nonlinear boundary value problems with multiple solutions. Nonlinear Anal. **47**, 1187–1198 (2001)
14. Hao, Z., Liu, L.: A necessary and sufficiently condition for the existence of positive solution of fourth-order singular boundary value problems. Appl. Math. Lett. **16**, 279–285 (2003)
15. Agarwal, R.P.: Focal Boundary Value Problems for Differential and Difference Equations. Kluwer Academic, Dordrecht (1998)
16. Agarwal, R.P., O'Regan, O., Wong, P.J.Y.: Positive Solutions of Differential, Difference, and Integral Equations. Kluwer Academic, Dordrecht (1998)
17. Zill, D.G., Cullen, M.R.: Differential Equations with Boundary-Value Problems, 5th edn. Brooks/Cole, Hong Kong (2001)
18. Graef, J.R., Yang, B.: On a nonlinear boundary value problem for fourth order equations. Appl. Anal. **72**, 439–448 (1999)
19. Wong, P.J.Y.: Triple positive solutions of conjugate boundary value problems. Comput. Math. Appl. **36**, 19–35 (1998)
20. Leray, J., Schauder, J.: Topologie et equations fonctionels. Ann. Sci. École Norm. Sup. **51**, 45–78 (1934)

Chapter 13
Study of the Problem of One-Dimensional Flow of Homogenous Fluids in Fractal Porous Media

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Abstract In this paper, for the first time the exact solution in the form of Mittag-Leffler series for the initial-boundary problem of the fractional differential equation is obtained expressing the process of one-dimensional motion in porous medium with complex permeability homogeneous fluid to gallery. The obtained result permits the theoretical calculations in the process of exploitation of oil fields with a fractal nature.

13.1 Introduction

In recent decades, in many areas of science and technology, the application of the theory of fractals, which depicts the pattern of disordered heterogeneous media, has intensively developed new areas of research and has begun to gain momentum. As it is known from the literature, application of fractal theory, whose geometric structure retains its basic properties when considered at different length scales, i.e. one has the property of self-similarity, has found its field of application, and is now widely used and developed in field of earth sciences, including solving oil field exploitation problems. The application of this method allows the development of general methods for modelling flow processes in porous media with complex inhomogeneous permeability and facilitates the evaluation of the effect of physical processes occurring in them.

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In porous medium having a fractal property the fluid flow rate dos not expressed with the help of law $v = -\frac{\kappa}{\mu}\frac{\partial p}{\partial x}$ as it usually occur in sand medium with a normal permeability, but obeys the following law, [1–5]

$$v = -\frac{\kappa}{\mu}\frac{\partial^\alpha p}{\partial x^\alpha}, \quad 0 < \alpha < 1. \tag{13.1}$$

If we substitute expression (13.1) into equation of the conservation law newline $\frac{\partial(m\rho)}{\partial t} = div(\rho v)$ then the determination of the unknown pressure function $p(x, t)$ is reduced to solving of the equation

$$\frac{\partial p}{\partial t} = \frac{\partial x}{\partial x}\left(\kappa\frac{\partial^\alpha p}{\partial x^\alpha}\right) \tag{13.2}$$

with suitable initial and boundary conditions.

Fractional differential equations also arise in medium with memory and in solve problems of motion of the fluid described by relaxation processes. In this case, the process of motion is described with the help of differential equation with fraction derivative with respect to time by real order, as

$$\frac{\partial^\beta p}{\partial t^\beta} = \kappa\frac{\partial^2 p}{\partial x^2}. \tag{13.3}$$

Equation (13.3) is called Nigmatullin's equation in the literature [8]. Here, β is a real number.

Without considering the physical nature of the process described in (13.3), this paper is devoted to solving the problem posed for an Eq. (13.3).

Let's consider the following problem

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\kappa\frac{\partial^\alpha p(x,t)}{\partial x^\alpha}\right) + f(x,t), \tag{13.4}$$

$$p(x, 0) = p_0(x), \tag{13.5}$$

$$p(a, t) = p_a(t), \quad p(b, t) = p_b(t). \tag{13.6}$$

Here, p_0, p_a and p_b are known functions, κ is a given constant positive number. $a \le x \le b$, $t \ge 0$, $a \ne 0$, $f(x, t)$ is the density function. It is assumed that inhomogeneous boundary condition (13.6) is brought to the homogeneous boundary condition by linear substitution $w(x, t) = \frac{p_b(t)-p_a(t)}{b-a}(x - a) + p_a(t)$. $\tilde{p}_0$ and $\tilde{\varphi}(x, \lambda)$ represent the initial and density functions obtained after this transformation, respectively.

In order to study the nature of the solution in detail of problem (13.4)–(13.6), let us first consider the stationary problem

$$\frac{\partial}{\partial x}\left(\kappa \frac{\partial^\alpha p}{\partial x^\alpha}\right) = 0. \tag{13.7}$$

From (13.7) we get

$$\frac{\partial^\alpha p}{\partial x^\alpha} = C_1. \tag{13.8}$$

Applying the operator

$$I^\alpha_{a+} p(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{a+}^{x} \frac{p(t)dt}{(x-t)^{1-\alpha}}$$

to both sides of Eq. (13.8), we get, [6, 7, 9, 10, 12]

$$p(x) = C_1 \frac{x^\alpha}{\Gamma(\alpha-1)} + C_2 \frac{x^{\alpha-1}}{(\alpha-1)!}. \tag{13.9}$$

The unknown C_1 and C_2 are determined from boundary conditions (13.6) as follows:

$$C_1 = \frac{\frac{1}{(\alpha-1)!}\left(p(a)b^{\alpha-1} - p(b)a^{\alpha-1}\right)}{\frac{a^{\alpha-1}b^{\alpha-1}}{\Gamma(\alpha-1)(\alpha-1)!}(a-b)}, \quad C_2 = \frac{\frac{1}{\Gamma(\alpha-1)}\left[a^\alpha p(b) - b^\alpha p(a)\right]}{\frac{a^{\alpha-1}b^{\alpha-1}}{\Gamma(\alpha-1)(\alpha-1)!}(a-b)}.$$

Substituting these expressions into (13.9), we obtain

$$p(x) = Ax^\alpha + Bx^{\alpha-1},$$

where

$$A = \frac{\frac{1}{(\alpha-1)!}\left(p(a)b^{\alpha-1} - p(b)a^{\alpha-1}\right)}{\frac{a^{\alpha-1}b^{\alpha-1}}{\Gamma(\alpha-1)(\alpha-1)!}(a-b)} \frac{1}{\Gamma(\alpha-1)},$$

$$B = \frac{\frac{1}{\Gamma(\alpha-1)}\left[a^\alpha p(b) - b^\alpha p(a)\right]}{\frac{a^{\alpha-1}b^{\alpha-1}}{\Gamma(\alpha-1)(\alpha-1)!}(a-b)} \frac{1}{(\alpha-1)!}.$$

It is known from the literature that, the mass of liquid run out through homogeneous medium of fractal nature to gallery is determined as follows:

$$Q = -\frac{\kappa f}{\mu} D^\alpha_{a+} p(x) = \frac{\kappa f}{\mu} \frac{p(b)a^{\alpha-1} - p(a)b^{\alpha-1}}{a^{\alpha-1}b^{\alpha-1}(a-b)} \Gamma(\alpha-1).$$

Note 13.1 From this formula, when $\alpha = 1$ we get the well-known Dupi formula:

$$A = \frac{p(a) - p(b)}{a - b}, \quad B = \frac{ap(b) - bp(a)}{a - b},$$

$$p(x) = \frac{p(a) - p(b)}{a - b} x^{\alpha} + = \frac{ap(b) - bp(a)}{a - b} x^{\alpha - 1},$$

$$Q = -\frac{\kappa f}{\mu} D^{\alpha}_{a+} p(x) = \frac{\kappa f}{\mu} \frac{p(b) - p(a)}{b - a}.$$

It should be noted that the solution of this issue is addressed in [11] with a different approach.

Now, let's try to find a solution to problem (13.4)–(13.6). For this purpose, applying the Laplace transform we get

$$\frac{d^{\alpha+1} \tilde{p}(x, \lambda)}{dx^{\alpha+1}} - \frac{\lambda}{\kappa} \tilde{p}(x, \lambda) = \frac{F(x, \lambda)}{\kappa}, \tag{13.10}$$

$$\tilde{p}(a, \lambda) = \tilde{p}(b, \lambda) = 0 \tag{13.11}$$

where $F(x, \lambda) = -\tilde{p}_0(x) + \tilde{\varphi}(x, \lambda)$ and

$$\tilde{p}(x, \lambda) \equiv \mathcal{L}(p(x, t)) = \int_0^{\infty} e^{-\lambda t} p(x, t) dt, \quad \tilde{f}(x, \lambda) \int_0^{\infty} e^{-\lambda t} f(x, t) dt.$$

Firstly, let us look for the solution of the following homogeneous equation

$$\frac{d^{\alpha+1} \tilde{p}(x, \lambda)}{dx^{\alpha+1}} - \frac{\lambda}{\kappa} \tilde{p}(x, \lambda) = 0, \tag{13.12}$$

corresponding to Eq. (13.10) in the form of the Mittag-Leffler function [6, 9, 10, 12]

$$\tilde{p}(x, \lambda) = \sum_{k=0}^{\infty} \rho^k \frac{x^{-1+(k+1)\frac{\alpha+1}{2}}}{\left(-1 + (k+1)\frac{\alpha+1}{2}\right)!}.$$

Here, ρ is an unknown parameter for now. Substituting this expression into (13.12) gives

$$\rho^2 \tilde{p}(x, \lambda) - \frac{\lambda}{\kappa} \tilde{p}(x, \lambda) = 0.$$

From here we get the characteristic equation

$$\rho^2 - \frac{\lambda}{\kappa} = 0. \tag{13.13}$$

From (13.13), $\rho_{1,2} = \pm\sqrt{\frac{\lambda}{\kappa}}$. Thus, we can write the general solution of the homogeneous equation in the form

$$\tilde{p}_h(x,\lambda) = C_1 \tilde{p}_h^{(1)}(x,\lambda) + C_2 \tilde{p}_h^{(2)}(x,\lambda), \tag{13.14}$$

where

$$\tilde{p}_h^{(1)}(x,\lambda) = \sum_{k=0}^{\infty} \left(\sqrt{\frac{\lambda}{\kappa}}\right)^k \frac{x^{-1+(k+1)\frac{\alpha+1}{2}}}{\left(-1+(k+1)\frac{\alpha+1}{2}\right)!},$$

$$\tilde{p}_h^{(2)}(x,\lambda) = \sum_{k=0}^{\infty} \left(-\sqrt{\frac{\lambda}{\kappa}}\right)^k \frac{x^{-1+(k+1)\frac{\alpha+1}{2}}}{\left(-1+(k+1)\frac{\alpha+1}{2}\right)!}$$

are the system of fundamental solutions of a homogeneous equation. Here, C_1 and C_2 are arbitrary constants.

As in the theory of ordinary differential equations, we will express the general solution of the inhomogeneous equation (13.10) as the sum of the general solution of the corresponding homogeneous equation and any special solution of it. For this purpose, to find the special solution of Eq. (13.10) we will write it in the following form

$$D^{\frac{\alpha+1}{2}}\left(D^{\frac{\alpha+1}{2}}\right)\tilde{p}(x,\lambda) - \frac{\lambda}{\kappa}\tilde{p}(x,\lambda) = 0, \tag{13.15}$$

and seek the solution in the form of the Mittag-Leffler series

$$\tilde{P}(x,\lambda) = \sum_{s=1}^{\infty} p_s \frac{x^{s\frac{\alpha+1}{2}-1}}{\left(s\frac{\alpha+1}{2}-1\right)!}. \tag{13.16}$$

Let us express the known function on the right hand side of the equation as the Mittag-Leffler function

$$\frac{F(x,\lambda)}{\kappa} = \sum_{s=1}^{\infty} f_s(\lambda) \frac{x^{s\frac{\alpha+1}{2}-1}}{\left(s\frac{\alpha+1}{2}-1\right)!}.$$

From (13.16)

$$D^{\frac{\alpha+1}{2}}\tilde{p}(x,\lambda) = \underbrace{p_1\frac{x^{-1}}{(-1)!}}_{=0} + p_2\frac{x^{\frac{\alpha+1}{2}-1}}{\left(\frac{\alpha+1}{2}-1\right)!} + p_3\frac{x^{2\frac{\alpha+1}{2}-1}}{\left(2\frac{\alpha+1}{2}-1\right)!}$$

$$+ p_4\frac{x^{3\frac{\alpha+1}{2}-1}}{\left(3\frac{\alpha+1}{2}-1\right)!} + p_5\frac{x^{4\frac{\alpha+1}{2}-1}}{\left(4\frac{\alpha+1}{2}-1\right)!} + p_6\frac{x^{5\frac{\alpha+1}{2}-1}}{\left(5\frac{\alpha+1}{2}-1\right)!} + \cdots$$

and

$$D^{\frac{\alpha+1}{2}}\left(D^{\frac{\alpha+1}{2}}\tilde{p}(x,\lambda)\right) = \underbrace{p_2\frac{x^{-1}}{(-1)!}}_{=0} + p_3\frac{x^{\frac{\alpha+1}{2}-1}}{\left(\frac{\alpha+1}{2}-1\right)!} + p_4\frac{x^{2\frac{\alpha+1}{2}-1}}{\left(2\frac{\alpha+1}{2}-1\right)!}$$

$$+ p_5\frac{x^{3\frac{\alpha+1}{2}-1}}{\left(3\frac{\alpha+1}{2}-1\right)!} + p_6\frac{x^{4\frac{\alpha+1}{2}-1}}{\left(4\frac{\alpha+1}{2}-1\right)!} + \cdots$$

are obtained. Substituting these expressions into (13.15) gives

$$p_3\frac{x^{\frac{\alpha+1}{2}-1}}{\left(\frac{\alpha+1}{2}-1\right)!} + p_4\frac{x^{2\frac{\alpha+1}{2}-1}}{\left(2\frac{\alpha+1}{2}-1\right)!} + p_5\frac{x^{3\frac{\alpha+1}{2}-1}}{\left(3\frac{\alpha+1}{2}-1\right)!} + p_6\frac{x^{4\frac{\alpha+1}{2}-1}}{\left(4\frac{\alpha+1}{2}-1\right)!} + \cdots$$

$$-\frac{\lambda}{\kappa}\left(p_1\frac{x^{\frac{\alpha+1}{2}-1}}{\left(\frac{\alpha+1}{2}-1\right)!} + p_2\frac{x^{2\frac{\alpha+1}{2}-1}}{\left(2\frac{\alpha+1}{2}-1\right)!} + p_3\frac{x^{3\frac{\alpha+1}{2}-1}}{\left(3\frac{\alpha+1}{2}-1\right)!}\right.$$

$$\left. + p_4\frac{x^{4\frac{\alpha+1}{2}-1}}{\left(4\frac{\alpha+1}{2}-1\right)!} + p_5\frac{x^{5\frac{\alpha+1}{2}-1}}{\left(5\frac{\alpha+1}{2}-1\right)!} + p_6\frac{x^{6\frac{\alpha+1}{2}-1}}{\left(6\frac{\alpha+1}{2}-1\right)!} + \cdots\right)$$

$$= f_1(\lambda)\frac{x^{\frac{\alpha+1}{2}-1}}{\left(\frac{\alpha+1}{2}-1\right)!} + f_2(\lambda)\frac{x^{2\frac{\alpha+1}{2}-1}}{\left(2\frac{\alpha+1}{2}-1\right)!} + f_3(\lambda)\frac{x^{3\frac{\alpha+1}{2}-1}}{\left(3\frac{\alpha+1}{2}-1\right)!}$$

$$+ f_4(\lambda)\frac{x^{4\frac{\alpha+1}{2}-1}}{\left(4\frac{\alpha+1}{2}-1\right)!} + f_5(\lambda)\frac{x^{5\frac{\alpha+1}{2}-1}}{\left(5\frac{\alpha+1}{2}-1\right)!} + f_6(\lambda)\frac{x^{6\frac{\alpha+1}{2}-1}}{\left(6\frac{\alpha+1}{2}-1\right)!} + \cdots .$$

From the last equation, for the unknown p_s coefficients, we get the expressions

$$p_{2k}(\lambda) = \sum_{q=1}^{k} \left(\frac{\lambda}{\kappa}\right)^{k-q} f_{2q}(\lambda), \quad p_{2k+1}(\lambda) = \sum_{q=1}^{k} \left(\frac{\lambda}{\kappa}\right)^{k-q} f_{2q-1}(\lambda), \; k = 2, 3, \ldots \tag{13.17}$$

under the condition $p_1 = p_2 = 0$. Considering Eq. (13.17), a special solution of Eq. (13.15) is found in the following form

$$\tilde{P}(x,\lambda) = \sum_{s=1}^{s=1} p_s(\lambda) \frac{x^{s\frac{\alpha+1}{2}-1}}{\left(s\frac{\alpha+1}{2}-1\right)!}. \tag{13.18}$$

According to the general theory, the solution of problem (13.10) and (13.11) is written as

$$\tilde{p}(x,\lambda) = C_1 \sum_{k=0}^{\infty} \left(\sqrt{\frac{\lambda}{\kappa}}\right)^k \frac{x^{-1+(k+1)\frac{\alpha+1}{2}}}{\left(-1+(k+1)\frac{\alpha+1}{2}\right)!}$$

$$+ C_2 \sum_{k=0}^{\infty} \left(-\sqrt{\frac{\lambda}{\kappa}}\right)^k \frac{x^{-1+(k+1)\frac{\alpha+1}{2}}}{\left(-1+(k+1)\frac{\alpha+1}{2}\right)!} + \sum_{s=1}^{s=1} p_s(\lambda) \frac{x^{s\frac{\alpha+1}{2}-1}}{\left(s\frac{\alpha+1}{2}-1\right)!}$$

or in short as

$$\tilde{p}(x,\lambda) = C_1 \tilde{p}_h^{(1)}(x,\lambda) + C_2 \tilde{p}_h^{(2)}(x,\lambda) + \tilde{P}(x,\lambda). \tag{13.19}$$

The unknown constants C_1 and C_2 are found from the following system of equations

$$\begin{cases} C_1 \tilde{p}_h^{(1)}(a,\lambda) + C_2 \tilde{p}_h^{(2)}(a,\lambda) = -\tilde{P}(a,\lambda), \\ C_1 \tilde{p}_h^{(1)}(b,\lambda) + C_2 \tilde{p}_h^{(2)}(b,\lambda) = -\tilde{P}(b,\lambda) \end{cases} \tag{13.20}$$

using boundary conditions (13.11). Substituting the obtained constants into (13.19), we get the solution of problem (13.10), (13.11) as follows

$$\tilde{p}(x,\lambda) = \frac{\Delta_1}{\Delta} \tilde{p}_h^{(1)}(x,\lambda) + \frac{\Delta_2}{\Delta} \tilde{p}_h^{(2)}(x,\lambda) + \tilde{P}(x,\lambda).$$

Here, Δ, Δ_1, Δ_2 are the main and auxiliary determinants consisting of the coefficients of the system (13.20), respectively.

Thus, we get the solution of problem (13.4)–(13.6) in the following form

$$p(x,t) = \mathcal{L}^{-1}\left(\tilde{p}(x,\lambda)\right).$$

Here, $\mathcal{L}^{-1}$ represents the inverse Laplace operator.

Note 13.2 Since the Laplace parameter in the function $\tilde{p}(x,\lambda)$ is included as a polynomial product, the original function is easily calculated if the initial data are known.

13.2 Conclusion

In this paper, for the first time, the exact solution of the boundary value problem for the $\alpha + 1$ $(0 < \alpha < 1)$ order fractal differential equation describing the one-dimensional flow process of a single-phase fluid in deposits with complex permeability is obtained in the form of Mittag-Leffler series.

References

1. Afonin, A.A., Sukhinov, A.I.: Mathematical models of geofiltrration and geomigration in porous media with fractal structure. Izv. YuFU Tekhntch. Nauk **97**(8), 62–70 (2009)
2. Barabanov, V.L.: Fractal model of the initial stage of capillary impregnation of rocks. Georesour. Geoenergy Geopolit. **1**(13), 1–16 (2016)
3. Belevtsov, N.S., Lukashchuk, S.Y.: Investigation of a fractional-differential model of single-phase filtration with Riesz potential. Multiphase Syst. **15**(1–2), 14–14 (2020)
4. Gazizov, R.K., Lukashchuk, S.Y.: Fractional-differential approach to modelling filtration processes in complex heterogeneous porous media. Vestnik Ufa State Aviation Tech. Univ. **21**(4), 104–112 (2017)
5. Kashchenko, N.M.: Fractal model of filtration in conditions of drainage operation. Bullet I. Kant Baltic Federal Univ. Phys. Math. Techn. Sci. **4**, 158–162 (2010)
6. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, North-Holland (2006)
7. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
8. Nigmatullin, R.R.: The realization of the generalized transfer equation in a medium with fractal geometry. Phys. Status Solidi **133**(1), 425–430 (1986)
9. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)
10. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Montreux (1993)
11. Suleymanov, B.A., Abbasov, E.M., Efendieva, A.O.: Stationary filtration in a fractally inhomogeneous porous medium. J. Eng. Phys. **78**(4), 194–196 (2005)
12. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)

Chapter 14
Nonlocal Initial-Boundary Value Problems for a Degenerate Hyperbolic Equation

Myrzagali Bimenov and Arailym Omarbaeva

Abstract The paper considers initial-boundary value problems for the degenerate hyperbolic equation $y^m u_{xx} - u_{yy} - b^2 y^m u = 0$ in the rectangular domain $\Omega = \{(x, y) : 0 < x < 1, 0 < y < T\}$, where $m > 0$, $b \geq 0$, $T > 0$ are given real numbers. We study problems with classical initial conditions $u(x, 0) = \tau(x)$, $u_y(x, 0) = \nu(x)$, $0 \leq x \leq 1$, and nonlocal boundary conditions $u_x(0, y) = \alpha u_x(1, y)$, $u(0, y) = u(1, y)$, or $u_x(0, y) = u_x(1, y)$, $u(0, y) = \beta u(1, y)$ with $0 \leq y \leq T$. Using the method of spectral analysis, we prove uniqueness and existence theorems for solutions of these problems.

14.1 Introduction

In the rectangular domain $\Omega = \{(x, y) : 0 < x < 1, 0 < y < T\}$ consider a degenerate hyperbolic equation

$$Lu \equiv y^m u_{xx} - u_{yy} - b^2 y^m u = 0, \tag{14.1}$$

where $m > 0$, $b \geq 0$, $T > 0$ are given real numbers. For this equation we will study the following nonlocal initial-boundary value problems.

Problem P_α. *Find a function* $u(x, y) \in C^1\left(\overline{\Omega}\right) \cap C^2(\Omega)$ *which is a solution of Eq.* (14.1) *in* Ω *and at the boundary satisfies the following classical initial*

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conditions

$$u(x,0) = \tau(x),\ u_y(x,0) = \nu(x),\ 0 \le x \le 1, \tag{14.2}$$

and the following nonlocal boundary conditions

$$u_x(0,y) = \alpha u_x(1,y),\ 0 \le y \le T, \tag{14.3}$$

$$u(0,y) = u(1,y),\ 0 \le y \le T, \tag{14.4}$$

where α is a given real number, and $\tau(x)$, $\nu(x)$ are given sufficiently smooth functions.

***Problem** P_β. Find a function $u(x,y) \in C^1\left(\overline{\Omega}\right) \cap C^2(\Omega)$ which is a solution of Eq. (14.1) in Ω and at the boundary satisfies the classical initial conditions (14.2) and the following nonlocal boundary conditions*

$$u_x(0,y) = u_x(1,y),\ 0 \le y \le T, \tag{14.5}$$

$$u(0,y) = \beta u(1,y),\ 0 \le y \le T, \tag{14.6}$$

where β is a given real number, and $\tau(x)$, $\nu(x)$ are given sufficiently smooth functions.

It is obvious that necessary conditions for these classes to belong to the class $u(x,y) \in C^1\left(\overline{\Omega}\right) \cap C^2(\Omega)$ are the fulfilment of natural conditions for matching initial and boundary values. For Problem P_α these are the conditions

$$\nu(0) = \alpha\nu(1),\ \tau(0) = \tau(1). \tag{14.7}$$

and for Problem P_β these are the conditions

$$\nu(0) = \nu(1),\ \tau(0) = \beta\tau(1). \tag{14.8}$$

In what follows we assume these conditions to be met. Moreover, we will require the fulfilment of additional matching conditions so that functional series, in the form of which a formal solution will be constructed, converge uniformly and all their derivatives of required order also converge uniformly.

The study of initial-boundary value problems with nonlocal boundary conditions has began relatively recently. One of the main research methods is the Fourier method (method of separation of variables). When using this method there arises a spectral problem for an ordinary differential operator. If boundary conditions of the obtained differential operator are self-adjoint, then this operator has a system of eigenfunctions that forms an orthonormal basis. Therefore, the solution of the original initial-boundary value problem can be constructed as an expansion using the obtained orthonormal system. This is the main idea of the Fourier method.

The situation is much more complicated when the boundary conditions of that spectral problem are not self-adjoint. Then this spectral problem may not even have a spectrum or the system of eigenfunctions may not have a basis property. In the case when the boundary conditions are strongly regular by Birkhoff, in the works of Keselman [1], Mikhailov [2] and Dunford and Schwartz [3] the Riesz basis property of the system of eigen- and associated functions of the spectral problem in $L_2(0, 1)$ was proved. In this case, this problem can have at most a finite number of associated functions. Based on this fact, Ionkin and Moiseev [4], using the method of separation of variables, constructed a solution of the initial-boundary value problem for a one-dimensional heat equation with strongly regular boundary conditions

$$\begin{cases} a_1 u_x(0,t) + b_1 u_x(1,t) + a_0 u(0,t) + b_0 u(1,t) = 0, \\ c_1 u_x(0,t) + d_1 u_x(1,t) + c_0 u(0,t) + d_0 u(1,t) = 0, \end{cases} \tag{14.9}$$

and proved its uniqueness and stability with respect to the initial data in various norms.

In the case when boundary conditions (14.9) are regular but not strongly regular, the system of root functions of arising spectral problem are complete and minimal but may not form an unconditional basis in $L_2(0, 1)$. And in that case, the use of the Fourier method is impossible and an additional research is required. In [5], a methodology was demonstrated for solving initial-boundary value problems for a heat equation with regular but not strongly regular boundary conditions of form (14.9). And in [6] this methodology was used to solve an inverse problem of recovering an unknown part of the heat equation with respect to initial and final conditions with regular but not strongly regular boundary conditions of form (14.9). The main idea of the methodology was to reduce the original problem to a sequential solution of two initial-boundary value problems with strongly regular boundary conditions.

Problems for differential equations of fixed type, for equations with a degenerate coefficient at the highest derivative have their own features when studied by the Fourier method. The main difficulty arises in justifying the convergence of functional series that represent formally constructed solutions of the problems. The use of nonlocal boundary conditions for degenerate equations strengthens these difficulties.

We present some previous works by other authors which are the closest to our research. Lerner and Repin [7] in the half-strip $D = \{(x, y)|0 < x < 1, y > 0\}$ studied the problem for a degenerate elliptic equations: find a function $u(x, y)$ with the properties

$$u(x, y) \in C(D) \cap C^1(D \cup \{x = 0\}) \cap C^2(D),$$

$$y^m u_{xx} + u_{yy} = 0, \quad (x, y) \in D, \quad m > -1,$$

$$u(x, y) \to 0 \text{ at } y \to +\infty \text{ uniformly with respect to } x \in [0, 1],$$

$$u(0, y) - u(1, y) = \phi_1(y), \quad u_x(0, y) = \phi_2(y), \quad y \geq 0,$$

$$u(x, 0) = \tau(x), \quad 0 \leq x \leq 1,$$

where $\tau(x)$, $\phi_1(y)$, $\phi_2(y)$ are given sufficiently smooth functions, and besides $\tau(x)$ is orthogonal to the system of functions $1, \cos((2n+1)\pi x),\ n = 0, 1, 2, \ldots$.

Also, in [8] a similar problem was studied in the half-strip D for an elliptic equation with a singular coefficient

$$u_{xx} + u_{yy} + \frac{2p}{y} u_y - b^2 u = 0, \quad b \geq 0, \quad p \in \mathbb{R},$$

in the case when $\phi_1(y) \equiv 0$and $\phi_2(y) \equiv 0$. The uniqueness of the solution was proven based on the extremum principle. The solvability of the problem under consideration was established using the method of separation of variables and the method of integral transformations.

Moiseev [9] studied a nonlocal boundary value problem in the half-strip D for a degenerate elliptic equation:

$$y^m u_{xx} + u_{yy} = 0, \quad m > -2,$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq 1,$$

$$u(0, y) = u(1, y), \quad u_x(0, y) = 0, \quad y \geq 0,$$

$$f(x) \in C^{2+\alpha}[0, 1], \quad f(0) = f(1), \quad f'(0) = 0$$

in the class of functions $u(x, y) \in C(D) \cap C^2(D)$ tending to zero and bounded at infinity. The uniqueness and existence of the solution were proved using the method of spectral analysis. Moreover, the solution of the problem was constructed in the form of a sum of biorthogonal series.

Later, in [10], these results were transferred to the equations

$$y^m u_{xx} + u_{yy} - b^2 y^m u = 0,\ b = \text{const} \geq 0,\ m > 0.$$

Sabitov and Sidorenko [11] for Eq. (14.9) in the domain D solved a problem with the initial conditions (4) and periodical boundary conditions. That is, they considered the problem P_α for a particular case $\alpha = 1$. The proof of the uniqueness and existence of the solution of the problem was also carried out using the spectral method. In [12] following [10, 11], existence and uniqueness theorems of solutions for Problems P_α and P_β were proved in particular cases when $\alpha = 0$ and $\beta = 0$, respectively. The solutions of the problems posed were constructed in the form of a sum of biorthogonal series.

In this work, we consider Problems P_α and P_β for the case of arbitrary constants α and β.

14.2 Problem P_α

The application of the Fourier method to Problem P_α leads to the consideration of the spectral problem for the ordinary differential operator

$$-X''(x) = \lambda X(x),\ 0 \le x \le 1, \tag{14.10}$$

with nonlocal boundary conditions

$$X'(0) = \alpha X'(1),\ X(0) = X(1). \tag{14.11}$$

Boundary conditions (14.11) are regular but not strongly regular [13]. In the case when $\alpha = 1$, this problem is self-adjoint and the system of its normalized eigenfunctions forms an orthonormal basis in $L_2(0, 1)$. This features was used in [11]. And in the case when $\alpha \neq 1$, this problem has an infinite number of associated functions [14]. And in that case if the system of root functions forms an unconditional basis, then the solution of the initial-boundary value problem can be constructed using the method of biorthogonal expansion. For the heat equation this method was first implemented by Ionkin [15], and for Eq. (14.1) in the work [12] for particular cases $\alpha = 0$ and $\beta = 0$.

All eigenvalues of problem (14.10) and (14.11) are double $\lambda_n = (2\pi n)^2$, except for a simple zero eigenvalue $\lambda_0 = 0$. Each double eigenvalue corresponds to one eigenfunction and one associated function [14]. The system of eigenfunctions X_{2n} and associated functions X_{2n-1} of problem (14.10)–(14.11) has the form

$$X_0(x) = 1,\ X_{2n} = \cos(2\pi n x),\ X_{2n-1}(x) = [\alpha + (1-\alpha)x]\sin(2\pi n x),\ \ n \in N. \tag{14.12}$$

They satisfy the equations

$$-X''_{2n}(x) = \lambda_n X_{2n}(x),\ 0 \le x \le 1, \tag{14.13}$$

$$-X''_{2n-1}(x) = \lambda_n X_{2n-1}(x) - 2(1-\alpha)(2\pi n)X_{2n}(x),\ 0 \le x \le 1. \tag{14.14}$$

It is proved that system (14.12) forms a Riesz basis in $L_2(0, 1)$. Therefore, the solution of Problem P_α can be represented in the form of a biorthogonal expansion with respect to system (14.12)

$$u(x, y) = u_0(y) + \sum_{n=1}^{\infty} u_n(y)\cos(2\pi nx) + \sum_{n=1}^{\infty} v_n(y)[\alpha + (1-\alpha)x]\sin(2\pi nx), \tag{14.15}$$

where the functions $u_0(y)$, $u_n(y)$, $v_n(y)$ must be defined from the differential equation (14.1) and from the initial conditions (14.2) of the problem.

For these functions, ordinary differential equations of the following form

$$v_n''(y) + y^m\left[(2\pi n)^2 + b^2\right]v_n(y) = 0,$$

$$u_0''(y) + b^2 y^m u_0(y) = 0,$$

$$u_n''(y) + \left[(2\pi n)^2 + b^2\right]y^m u_n(y) = 4\pi n(1-\alpha)y^m v_n(y)$$

with the corresponding initial data at the point $y = 0$ are obtained.

These problems (Cauchy problems) have unique solutions that can be constructed explicitly through the Bessel functions of the first and second kind. Thus, a formal solution of Problem P_α is constructed in the form of series (14.15). Requiring smoothness and matching conditions from the initial data $\tau(x), \nu(x)$, we have proved the uniform convergence of the functional series (14.15) and possibility of its term-be-term differentiation the required number of times.

These are complicated and cumbersome calculations, we will not dwell on them here.

Let us formulate the main result on the well-posedness of Problem P_α.

Theorem 14.1 *Let $\alpha \neq 1$. If $\tau(x), \nu(x) \in C^3[0, 1]$ and satisfy the matching conditions*

$$\tau(0) = \tau(1),\ \tau'(0) = \alpha\tau'(1),\ \tau''(0) = \tau''(1), \tag{14.16}$$

$$\nu(0) = \nu(1),\ \nu'(0) = \alpha\nu'(1),\ \nu''(0) = \nu''(1), \tag{14.17}$$

then there exists a unique solution of Problem P_α and can be represented in the form of a sum of series (14.15)*, where the functions $u_0(y)$, $u_n(y)$, $v_n(y)$ are uniformly defined from the differential equation* (14.1) *and the initial conditions* (14.2) *of the problem.*

In the case when $\alpha = 1$, Problem P_α is not Noetherian since the corresponding homogeneous initial-boundary value problem has an infinite number of linearly independent solutions.

14.3 Problem P_β

The boundary conditions in Problem P_β are also regular but not strongly regular. The spectral problem, arising when applying the method of separation of variables, has an infinite number of double eigenvalues. Each of them corresponds to one eigenfunction and one associated function. These associated functions can be chosen in such a way that the entire system of root functions forms the Riesz basis. And therefore this system can be used to implement the Fourier method.

Let us immediately formulate the main result on the well-posedness of Problem P_β.

Theorem 14.2 *Let $\beta \neq 1$. If $\tau(x), \nu(x) \in C^3[0, 1]$ and satisfy the matching conditions*

$$\tau(0) = \beta\tau(1),\ \tau'(0) = \tau'(1),\ \tau''(0) = \beta\tau''(1), \tag{14.18}$$

$$\nu(0) = \beta\nu(1),\ \nu'(0) = \nu'(1),\ \nu''(0) = \beta\nu''(1), \tag{14.19}$$

then there exists a unique solution of Problem P_β and it can be represented in the form of a sum of uniformly convergent functional series.
In the case when $\beta = 1$, Problem P_β is not Noetherian since the corresponding homogeneous initial-boundary value problem has an infinite number of linearly independent solutions.

Note that various types of nonlocal boundary value and initial-boundary value problems are actively studied for various types of differential equations. We will give only some references [16–29] to recent works, the topics of which are quite close to the topic of our research.

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References

1. Keselman, G.M.: On the unconditional convergence of eigenfunction expansions of certain differential operators. Izv. Vyssh. Uchebn. Zaved. Mat. **2**, 82–93 (1964)
2. Mikhailov, V.P.: On Riesz basis in $L_2(0, 1)$. Dokl. Akad. Nauk SSSR. **144**(5), 981–984 (1962)
3. Dunford, N., Schwartz, J.T.: Linear Operators. Part III, Spectral Operators. Wiley, New York (1971)
4. Ionkin, N.I., Moisseev, E.I.: A problem for the heat equation with two-point boundary conditions. Differ. Equ. **15**, 1284–1295 (1979)
5. Sadybekov, M.A.: Initial-boundary value problem for a heat equation with not strongly regular boundary conditions. In: Functional Analysis in Interdisciplinary Applications. Springer Proceedings in Mathematics & Statistics, vol. 216, pp. 330–348. Springer, Berlin (2017)

6. Orazov, I., Sadybekov, M.A.: On a class of problems of determining the temperature and density of heat sources given initial and final temperature. Siberian Math. J. **53**(1), 146–151 (2012)
7. Lerner, M.E., Repin, O.A.: On Frankl-type problems for some elliptic equations with degeneration of various types. Differ. Uravn. **35**(8), 1087–1093 (1999)
8. Lerner, M.E., Repin, O.A.: Nonlocal boundary value problems in a vertical half-strip for the generalized axisymmetric Helmholtz equation. Differ. Uravn. **37**(11), 1562–1564 (2001)
9. Moiseev, E.I.: On the solution of a nonlocal boundary value problem by the spectral method. Differ. Uravn. **35**(8), 1094–1100 (1999)
10. Moiseev, E.I.: On the solvability of a nonlocal boundary value problem. Differ. Uravn. **37**(11), 1565–1567 (2001)
11. Sabitov, K.B., Sidorenko, O.G.: Solvability of a nonlocal boundary-value problem. In: Proceedings of All-Russia Conference -Contemporary Problems in Physics and Mathematics - (Ufa), vol. 1, pp. 80–86 (2004)
12. Sabitova, Y.K.: Nonlocal initial-boundary-value problems for a degenerate hyperbolic equation. Russ. Math. **53**, 41–49 (2009)
13. Naimark, M.A.: Linear Differential Operators: Elementary Theory of Linear Differential Operators. Frederick Ungar, New York (1967)
14. Lang, P., Locker, J.: Spectral theory of two-point differential operators determined by $-D^2$. J. Math. Anal. Appl. **146**(1), 148–191 (1990)
15. Ionkin, N.I.: Solution of a boundary-value problem in heat conduction with a non-classical boundary condition. Differ. Equ. **13**, 204–211 (1977)
16. Sabitova, Y.K.: The first boundary problem with an integral condition for a mixed-type equation with a characteristic degeneration. Russ. Math. **64**(11), 39–57 (2020)
17. Karimov, K.T.: Nonlocal problem for a three-dimensional elliptic equation with singular coefficients in a rectangular parallelepiped. J. Sib. Fed. Univ. Math. Phys. **13**(5), 533–546 (2020)
18. Sabitov, K.B.: Asymptotic estimates of the difference of products of Bessel functions by the integral of these functions. Vestn. Samar. Gos. Tekh. Univ. Ser. Fiz.-Mat. Nauki. **24**(1), 41–55 (2020)
19. Urinov, A.K, Karimov, K.T.: Nonlocal boundary value problems for a three-dimensional elliptic equation with singular coefficients in a semi-infinite parallelepiped. Sib. Elektron. Mat. Izv. **17**, 161–178 (2020)
20. Sabitova, Y.K.: Dirichlet problem for mixed type equation with characteristic degeneration. Vestn. Samar. Gos. Tekh. Univ. Ser. Fiz.-Mat. Nauki. **23**(4), 622–645 (2019)
21. Assanova, A.T.: An integral-boundary value problem for a partial differential equation of second order. Turk. J. Math. **43**(4), 1967–1978 (2019)
22. Asanova, A.T.: One approach to the solution of a nonlocal problem for systems of hyperbolic equations with integral conditions. J. Math. Sci. **238**(3), 189–206 (2019)
23. Sabitov, K.B., Zaitseva, N.V.: Initial value problem for B-hyperbolic equation with integral condition of the second kind. Differ. Equ. **54**(1), 121–133 (2018)
24. Assanova, A.T.: Nonlocal problem with integral conditions for a system of hyperbolic equations in characteristic rectangle. Russ. Math. **61**(5), 7–20 (2017)
25. Zaitseva, N.V.: Keldysh type problem for B-hyperbolic equation with integral boundary value condition of the first kind. Lobachevskii J. Math. **38**(1), 162–169 (2017)
26. Sabitova, Y.K.: Boundary-value problem with nonlocal integral condition for mixed-type equations with degeneracy on the transition line. Math. Not. **98**(3), 454–465 (2015)
27. Orazov, I., Sadybekov, M.A.: One nonlocal problem of determination of the temperature and density of heat sources. Russ. Math. **56**(2), 60–64 (2012)
28. Sadybekov, M., Dildabek, G., Ivanova, M.: Direct and inverse problems for nonlocal heat equation with boundary conditions of periodic type. Bound. Value Probl. **2022**, 53 (2022)
29. Sadybekov, M., Dukenbayeva, A.: On boundary value problems of the Samarskii-Ionkin type for the Laplace operator in a ball. Complex Variables Elliptic Equ. **67**(2), 369–383 (2020)

Chapter 15
On Well-Posedness of the Nonlocal Boundary Value Problem with Samarskii-Ionkin Conditions for the 2m-th Order Multidimensional Elliptic Equations

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Abstract This paper investigates the nonlocal boundary value problem for 2m-th order multidimensional elliptic equations with Samarskii–Ionkin condition. The well-posedness of this problem in Hölder spaces is established.

15.1 Introduction

Elliptic partial differential equations have applications in almost all areas of mathematics, from harmonic analysis to geometry to Lie theory, as well as numerous applications in physics and engineering. The well-posedness of the several local boundary value problems for the elliptic equations and its related applications have been investigated by many researchers (see, for example, [1–6], and the references given therein).

In paper [7], Ionkin studied a nonlocal problem for a one-dimensional parabolic equation arising in modeling some nonclassical thermal processes. The existence of solutions was proved and, later, their stability was established in [8]. For a parabolic equation in one space variable, Samarskii [9] proposed a nonlocal boundary value problem formulation covering both classical initial-boundary value problems and Ionkin's problem from [7, 8]. Recently, various nonlocal boundary value problems with Samatskii-Ionkin condition for partial differential equations have been investigated by many researchers (see, e.g.[10–22], and the references given therein).

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In the present paper, we consider the nonlocal boundary value problem for $2m$-th order multidimensional elliptic equations with Samarskii–Ionkin condition. The well-posedness of this problem in Hölder spaces is established.

15.2 The Well-Posedness of the Nonlocal Boundary Value Elliptic Problem with Samarskii–Ionkin Condition

On the range $\{(x, y) : 0 \le y \le T, x \in \mathbb{R}^n\}$ the nonlocal boundary value problem

$$\begin{cases} -\frac{\partial^2 u}{\partial y^2} + \sum\limits_{|r|=2m} a_r(x) \frac{\partial^{|r|} u}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} + \delta u(y, x) = f(y, x), \\ 0 < y < T,\ x, r \in \mathbb{N}_0^n, |r| = r_1 + \cdots + r_n, \\ u(0, x) = \varphi(x),\ \frac{\partial u(0,x)}{\partial y} = \frac{\partial u(T,x)}{\partial y} + \int\limits_0^T \alpha(s) u(s, x) ds + \psi(x), x \in \mathbb{R}^n \end{cases} \tag{15.1}$$

for $2m-$order multidimensional elliptic equations with integral type Samarskii–Ionkin condition is considered. Here, $a_r(x)$ and $f(y, x)$, $\varphi(x)$, $\psi(x)$ are given sufficiently smooth functions and $\alpha(x) > 0$, $\delta > 0$ is the sufficiently large number. Let us consider a differential operator with constant coefficients of the form

$$B = \sum_{|r|=2m} b_r \frac{\partial^{|r|}}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}},$$

acting on functions defined on the entire space $\mathbb{R}^n$. Here, $r \in \mathbb{R}^n$ is a vector with nonnegative integer components, $|r| = r_1 + \cdots + r_n$. If $\varphi(y)$ $(y = (y_1, \ldots, y_n) \in \mathbb{R}^n)$ is an infinitely differentiable function that decays at infinity together with all its derivatives, then by means of the Fourier transformation one establishes the equality

$$F(B_\varphi)(\xi) = B(\xi) F(\varphi)(\xi).$$

Here, the Fourier transform operator is defined by the rule

$$F(\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} \exp\{-i (z, \xi)\} \varphi(z) dz,$$

where

$$(z, \xi) = \sum_{k=1}^{k} z_k \xi_k.$$

The function $B(\xi)$ is called the symbol of the operator B and is given by formula

$$B(\xi) = \sum_{|r|=2m} b_r (i\xi_1)^{r_1} \dots (i\xi_n)^{r_n} .$$

We will assume that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x) (i\xi_1)^{r_1} \dots (i\xi_n)^{r_n} , \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \tag{15.2}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1|\xi|^{2m} \le (-1)^m B^x(\xi) \le M_2|\xi|^{2m} < \infty$$

for $\xi \neq 0$. Here, $M_1 > 0$ and $M_2| < \infty$. Problem (15.1) has a unique smooth solution.

To formulate our results, we introduce the Banach space $C^\alpha(E)$, it is obtained by completion of the set of all smooth E-valued functions $\varphi(y)$ on $[0, T]$ in the norm

$$\|\varphi\|_{C^\alpha(E)} = \|\varphi\|_{C(E)} + \sup_{0 \le y < y+\tau \le T} \frac{\| \varphi(y+\tau) - \varphi(\tau) \|_E}{\tau^\alpha},$$

where $C(E)$ stands for the Banach space of all continuous functions $\varphi(y)$ defined on $[0, T]$ with values in E equipped with the norm

$$||\varphi||_{C(E)} = \max_{y \in [0,T]} \|\varphi(y)\|_E.$$

Moreover, we introduce the Banach space $C^\mu(\mathbb{R}^n)$ $(0 < \mu < 1)$ of all continuous functions $\varphi(x)$ defined on $\mathbb{R}^n$ and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{C^\mu(\mathbb{R}^n)} = \|\varphi\|_{C(\mathbb{R}^n)} + \sup_{x,z \in \mathbb{R}^n, z \neq 0} \frac{|\varphi(x+z) - \varphi(x)|}{|z|^\mu},$$

where $C(\mathbb{R}^n)$ is the space of all continuous functions $\varphi(x)$ defined on $\mathbb{R}^n$ with the usual norm

$$\|\varphi\|_{C(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |\varphi(x)| .$$

Now, let us state the following well-posedness result.

Theorem 15.1 *For the solution of the boundary value problem (15.1), the following coercivity inequalities are satisfied*

$$\|u\|_{C^{2+\alpha}\left(C^{\mu}(\mathbb{R}^n)\right)} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|}u}{\partial x_1^{r_1}\dots\partial x_n^{r_n}} \right\|_{C^{\alpha}\left(C^{\mu}(\mathbb{R}^n)\right)}$$

$$+ \|u\|_{C^{2}\left(C^{2m\alpha+\mu}(\mathbb{R}^n)\right)} \leq M(\alpha) \left[\|f\|_{C^{\alpha}\left(C^{\mu}(\mathbb{R}^n)\right)} \right.$$

$$+ \sum_{|r|=2m} \left\| a_r(\cdot)\frac{\partial^{|\tau|}\psi(\cdot)}{\partial x_1^{r_1}\dots\partial x_n^{r_n}} \right\|_{C^{m\alpha+\mu}(\mathbb{R}^n)}$$

$$\left. + \left\| \sum_{|r|=2m} a_r(\cdot)\frac{\partial^{|\tau|}\varphi(\cdot)}{\partial x_1^{r_1}\dots\partial x_n^{r_n}} + \delta\varphi(\cdot) - f(0,\cdot) + f(T,\cdot) \right\|_{C^{2m\alpha+\mu}(\mathbb{R}^n)} \right],$$

$$\|u\|_{C^{2}\left(C^{2m\alpha+\mu}(\mathbb{R}^n)\right)} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|}u}{\partial x_1^{r_1}\dots\partial x_n^{r_n}} \right\|_{C\left(C^{2m\alpha+\mu}(\mathbb{R}^n)\right)}$$

$$\leq M(\alpha) \left[\|f\|_{C\left(C^{2m\alpha+\mu}(\mathbb{R}^n)\right)} + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|}\varphi}{\partial x_1^{r_1}\dots\partial x_n^{r_n}} \right\|_{C\left(C^{2m\alpha+\mu}(\mathbb{R}^n)\right)} \right.,$$

$$\left. + \sum_{|\tau|=2m} \left\| \frac{\partial^{|r|}\psi}{\partial x_1^{r_1}\dots\partial x_n^{r_n}} \right\|_{C\left(C^{m\alpha+\mu}(\mathbb{R}^n)\right)} \right] 0 < 2m\alpha + \mu < 1$$

where $M(\alpha)$ does not depend on $\varphi(x)$, $\psi(x)$ and $f(y,x)$.

Proof Problem (15.1) can be written in abstract form

$$-u''(y) + Au(y) = f(y), 0 < y < T,$$

$$u(0) = \varphi,\ u'(0) = u'(T) + \int_0^T \alpha(s)\,u(s)ds + \psi. \tag{15.3}$$

in a Banach space $E = C(\mathbb{R}^n)$ with a strongly positive operator $A^x = B^x + \delta I$ defined by (15.2). The proof of Theorem 15.1 is based on the positivity of the operator A^x in $C^{\mu}(\mathbb{R}^n)$ on the following abstract Theorems, the structure of the fractional spaces $E_{\alpha}((A^x)^{\frac{1}{2}}, C(\mathbb{R}^n))$, and the coercivity inequality for an elliptic operator A^x in $C^{\mu}(\mathbb{R}^n)$.

□

Theorem 15.2 *Suppose $A\varphi - f(0) \in E_\alpha$, $A^{\frac{1}{2}}\psi \in E_\alpha$, $f(y) \in C^\alpha(E)(0 < \alpha < 1)$. If A is the positive operator in Banach space E, then boundary value problem (15.3) is well-posed in Hölder space $C^\alpha(E)$. For the solution $u(t)$ in $C^\alpha(E)$ of the boundary value problem, the coercive inequality*

$$\|u''\|_{C^\alpha(E)} + \|Au\|_{C^\alpha(E)} + \|u''\|_{C(E_\alpha)} \tag{15.4}$$

$$\leq \frac{M}{\alpha(1-\alpha)} \parallel f \parallel_{C^\alpha(E)} + \frac{M}{\alpha}[\|A\varphi - f(0)\|_{E_\alpha} + ||A^{\frac{1}{2}}\psi||_{E_\alpha}]$$

holds, where M does not depend on α, φ, ψ and $f(t)$. Here, the Banach space $E_\alpha = E_\alpha(B, E)$ $(0 < \alpha < 1)$ consists of those $v \in E$ for which the norm

$$\|v\|_{E_\alpha} = \sup_{z>0} z^{1-\alpha} \|Bexp\{-zB\}v\|_E + \|v\|_E$$

is finite.

Theorem 15.3 *Let A be the positive operator in a Banach space E and $A\varphi$, $A^{\frac{1}{2}}\psi \in E_\alpha$, $f(t) \in C(E_\alpha)$ $(0 < \alpha < 1)$ Then, for the solution $u(t)$ in $C(E_\alpha)$ of boundary value problem (15.3), the coercive inequality*

$$\left\|u''\right\|_{C(E_\alpha)} + \parallel Au \parallel_{C(E_\alpha)} \tag{15.5}$$

$$\leq M[\|A\varphi\|_{E_\alpha} + ||A^{\frac{1}{2}}\psi||_{E_\alpha} + \alpha^{-1}(1-\alpha)^{-1} \parallel f \parallel_{C(E_\alpha)}]$$

holds, where M does not depend on α, φ, ψ and $f(t)$.

The proof of Theorems 15.2 and 15.3 are based on the formulas

$$u(t) = (I - e^{-2TB})^{-1}\Big\{(e^{-tB} - e^{-(2T-t)B})\varphi + (e^{-(T-t)B} - e^{-(T+t)B})u\,(T)$$

$$-(e^{-(T-t)B} - e^{-(T+t)B})(2B)^{-1}\int_0^T (e^{-(T-s)B} - e^{-(T+s)B})f(s)ds\Big\}$$

$$+ (2B)^{-1}\int_0^T (e^{-|t-s|B} - e^{-(t+s)B})f(s)ds,$$

$$u\,(T) = -Q\left\{\left[\left(I - e^{-TB}\right)^2 + B^{-1}\int_0^T (e^{-sB} - e^{-(2T-s)B})\alpha\,(s)\,ds\right]\varphi\right.$$

$$- B^{-1}\int_0^T \left(e^{-sB} - e^{-(2T-s)B}\right) f(s)ds - B^{-1}\int_0^T (e^{-(T-s)B} - e^{-(T+s)B})f(s)ds$$

$$-\,(2B)^{-1}\int_0^T (e^{-(T-s)B}-e^{-(T+s)B})\alpha\,(s)\,ds$$

$$\times\, B^{-1}\int_0^T (e^{-(T-y)B}-e^{-(T+y)B})f(y)dy$$

$$+(I-e^{-2TB})\Bigg[(2B)^{-1}\int_0^T \alpha\,(s)\,B^{-1}$$

$$\int_0^T (e^{-|s-y|B}-e^{-(s+y)B})f(y)dyds+B^{-1}\psi\Bigg]\Bigg\},$$

where

$$B=A^{\frac{1}{2}},\,Q=\left(\left(I-e^{-TB}\right)^2+B^{-1}\int_0^T (e^{-(T-s)B}-e^{-(T+s)B})\alpha\,(s)\,ds\right)^{-1},$$

and the estimates

$$\|\exp\,(-tB)\|_{E\to E}\,,\,\|tB\exp(-tB)\|_{E\to E}\le M(B)e^{-\alpha(B)t}\;(t>0),$$

$$\|Q\|_{E\to E}\le M(B),$$

and the definition of the space E_α.

Theorem 15.4 ([4, 5]) $E_\alpha((A^x)^{\frac{1}{2}},C(\mathbb{R}^n))=C^{2m\alpha}(\mathbb{R}^n)$ *for all* $0<\alpha<\frac{1}{2m},0<\alpha<\frac{1}{2m}$.

Theorem 15.5 ([6]) *For the solution of the elliptic differential equation*

$$Au(x)=w(x),\;x\in\mathbb{R}^n$$

the following coercivity inequality holds

$$\sum_{|\tau|=2m}\left\|\frac{\partial^{|r|}u}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}\right\|_{C^\beta(\mathbb{R}^n)}\le M\,(\beta)\,\|w\|_{C^\beta(\mathbb{R}^n)}\,,0<\beta<1.$$

15.3 Conclusion

In this article, we study nonlocal boundary value problem for $2m$-th order multidimensional elliptic equations with integral type Samarskii–Ionkin condition. The well-posedness of this problem in Hölder spaces is proved. A high order of accuracy two-step difference schemes for the numerical solution of this differential problem with integral type Samarskii–Ionkin condition will be presented. Note that operator method of [1] will permit us to establish the stability of these difference schemes.This is a joint work with Allaberen Ashyralyev (Bahcesehir University, Istanbul, Turkiye).

References

1. Ashyralyev, A., Sobolevskii, P.E.: New Difference Schemes for Partial Differential Equations. Birkhäuser Verlag, Basel (2004)
2. Skubachevskii, A.L.: Elliptic Functional Differential Equations and Applications. Birkhäuser Verlag, Basel (1997)
3. Lunardi, A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhäuser Verlag, Basel (1995)
4. Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators. North-Holland, Amsterdam (1978)
5. Ashyralyev, A., Sobolevskii, P.E.: Well-Posedness of Parabolic Difference Equations. Birkhäuser Verlag, Basel (1994)
6. Ashyralyev, A., Hamad, A.: A note on fractional powers of strongly positive operators and their applications. Fract. Calc. Appl. Anal. **22**(2), 302–325 (2019)
7. Ionkin, N.I.: Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition. Differ. Uravn. **13**(2), 294–304 (1977)
8. Ionkin, N.I.: On the stability of a problem in heat conduction with a nonclassical boundary condition. Differ. Uravn. **15**(7), 1279–1283 (1979)
9. Samarskii, A.A.: On certain problems in the theory of differential equations. Differ. Uravn. **16**(11), 1925–1935 (1980)
10. Sadybekov, M.A., Torebek, B.T., Yessirkegenov, N.A.: On an analog of Samarskii–Ionkin type boundary value problem for the Poisson equation in the disk. AIP Conf. Proc. **1676**, 020035 (2015). https://doi.org/10.1063/1
11. Sadybekov, M.A., Yessirkegenov, N.A.: Spectral properties of a Laplace operator with Samarskii–Ionkin type boundary conditions in a disk. AIP Conf. Proc. **1759**, 020139 (2016). https://doi.org/10.1063/1
12. Sadybekov, M.A., Turmetov, B.K., Torebek, B.T.: Solvability of nonlocal boundary-value problems for the Laplace equation in the ball. Electron. J. Differ. Equ. **2014**, 157 (2014)
13. Sadybekov, M.A., Dukenbayeva, A.A., Yessirkegenov, N.A.: On a generalised Samarskii–Ionkin type problem for the Poisson equation. In: Ibragimov, Z., Levenberg, N., Rozikov, U, Sadullaev, A. (eds.), Algebra, Complex Analysis, and Pluripotential Theory. Springer Proceedings in Mathematics & Statistics. Springer, Berlin (2018)
14. Dukenbayeva, A.A.: On a generalised Samarskii–Ionkin type problem for the Poisson equation in the disk. Matematicheskiy Zhurnal **18**(1), 78–87 (2018)
15. Turmetov, B.K., Koshanova, M., Usmanov, K.: About solvability of some boundary value problems for Poisson equation in the ball conditions. Filomat **32**(3), 939–946 (2018)
16. Turmetov, B.K., Karachik, V.V.: On solvability of some boundary value problems for a biharmonic equation with periodic conditions. Filomat **32**(3), 947–953 (2018)
17. Karachik, V.V., Turmetov, B.K.: On solvability of some nonlocal boundary value problems for biharmonic equation. Math. Slovaca **70**(2), 329–342 (2020)
18. Turmetov, B.K.: Generalization of the Robin problem for the Laplace equation. Differ Equ. **55**(9), 1134–1142 (2019)
19. Sadybekov, M.A., Dukenbayeva, A.A.: On boundary value problem of the Samarskii–Ionkin type for the Laplace operator in a ball. Kazakh Math J. **20**(1), 84–94 (2020)

20. Il'in, V.A., Kritskov, L.V.: Properties of spectral expansions corresponding to non-self-adjoint differential operators. J. Math. Sci. **116**(5), 3489–3550 (2003)
21. Sadybekov, M.A., Dukenbayeva, A.A.: Direct and inverse problems for the Poisson equation with equality of flows on a part of the boundary. Int. J. Complex Variables Elliptic Equ. **64**(5), 777–791 (2019)
22. Sadybekov, M.A., Dukenbayeva, A.A.: On boundary value problems of the Samarskii–Ionkin type for the Laplace operator in a ball. Int. J. Complex Variables Elliptic Equ. **67**(2), 369–383 (2022)

Chapter 16
Smoothness for Degenerate Elliptic Equations with Matrix Weights

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Abstract We establish local boundedness, Harnack inequality and local regularity for weak solutions of quasilinear degenerate elliptic equations in divergence form. The degeneracy arises from a non negative, symmetric, measurable matrix-valued function Q(x) and two suitable non negative weight functions. Regularity results are achieved under minimal assumptions on the coefficient.

16.1 Introduction

The investigation conducted in [9] focuses on triple degenerate quasilinear elliptic equations, presented as follows:

$$-\operatorname{div}\left(h(x)\left|\sqrt{Q(x)}\nabla u\right|^{p-2}Q(x)\nabla u\right) = m(x)|f(x)|^{p-2}f(x)\,. \tag{16.1}$$

This kind of equations involves essential components: a non-negative definite, symmetric, measurable matrix weight function $Q(x)$ at its core and a non-negative weight h. The equation includes also a weight function m in its right-hand side.

In [9], we showed local boundedness for weak solutions and the Harnack inequality for positive weak solutions to (16.1). These achievements are contingent upon specific conditions imposed on f, in particular the belonging to a Stummel-Kato class, which implies continuity for weak solutions. A critical aspect of our previous investigation involved a sub-representation formula yielding embedding results crucial in controlling lower-order terms characterized by strong degeneracy and singularity [1–3, 5].

It is noteworthy that prior studies, notably by B. Franchi, C. Perez, and R.L. Wheeden in [10], shed light on scenarios where representation formulas cannot be

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applicable, often due to the geometrical setting and the validity of (1–1) Poincaré inequality.

Here, we consider (16.1) under a different approach (see also [4, 6–8]). We assume a (1-p) Poincaré inequality for some $p > 1$, that—as in [10]—allows us a different representation formula in terms of a special chain of balls. By using the new representation formula we are able to prove a Fefferman–Phong embedding result involving the Stummel-Kato class modeled to our setting.

Then, we perform an iteration scheme along the classical pattern drawn by Serrin in [11] and obtain our fundamental results, i.e. local boundedness of weak solutions and Harnack inequality for positive weak solutions.

Finally, we get continuity for weak solutions to Eq. (16.1) under Stummel-Kato assumption on f, while Hölder continuity when f belongs to Morrey spaces.

16.2 Main Results

Let Ω be a bounded domain in $\mathbb{R}^n$ endowed with a metric ρ. We denote by $B = B(x, r)$ the ball centered in x of radius r.

Let μ be a given measure and $Q : \Omega \to S_n$ be a μ-measurable matrix function taking values in S_n, the collection of all non-negative definite $n \times n$ symmetric matrices. We always assume that the p^{th} power of the pointwise operator norm of $\sqrt{Q}$ is a weight i.e. $v = \|Q(x)\|_{op}^{p/2} \in L^1_{loc}(\Omega; \mu)$ where

$$\|Q(x)\|_{op} := \sup_{\xi \in \mathbf{R}^n, |\xi|=1} \langle \xi, Q(x)\xi \rangle.$$

We consider Eq. (16.1) where h and m are $L^1_{loc}(\Omega)$ and f belongs to a Stummel-Kato class of functions defined below.

We work in an axiomatic setting assuming some geometric and analytic statements listed below.

(H1) *Segment property.* For every $x, y \in \Omega$, there exists a continuous curve $\gamma : I \subset \mathbb{R} \to \Omega$ connecting x, y such that $\rho(\gamma(s), \gamma(t)) = |s - t|$ for every $s, t \in I$.

(H2) We assume the σ-finite measures $d\mu = h\,dx$, $d\nu = m\,dx$ are both doubling with constants $A_\nu, A_\mu > 0$ respectively. Moreover we assume μ to be reverse-doubling of order 1, i.e. there exists a constant $C > 0$ such that for every pair of balls $\tilde{B} \subset B \Subset \Omega$ one has

$$\mu(B) \geq C \left(\frac{r(B)}{r(\tilde{B})}\right) \mu(\tilde{B}).$$

(H3) We assume $d\mu \ll d\nu$, $\nu(E) > 0$ for every Lebesgue measurable set $E \subset \Omega$ having positive Lebesgue measure and we assume that there exists $C > 0$

such that for every compact subset $E \subset \Omega$ we have

$$\int_E v\, d\mu \le C\nu(E)\,.$$

(H4) *Sobolev inequality.* Let $P > 1$ We assume there exist two constants $C_0 > 0$ and $k > 1$ such that given any compact subset E of Ω there exists a $\delta > 0$ so that for any ball B with a radius r less than δ one has

$$\left(\frac{1}{\nu(B)}\int_B |u-u_B|^{kp} d\nu\right)^{\frac{1}{kp}} \le C_0 r\left(\frac{1}{\mu(B)}\int_B |\sqrt{Q}\nabla u|^p d\mu\right)^{\frac{1}{p}}$$

for any $u \in Lip(B)$, the collection of all Lipschitz functions defined on B, where $u_B = \frac{1}{\nu(B)}\int_B u d\nu$.

(H5) *Accumulating sequences of Lipschitz cut-off functions.* We assume that given any metric ball $B = B(x,r)$ such that $B \Subset \Omega$, there exists a sequence of functions $\varphi_j \in Lip_0(B)$ so that for every $j \in \mathbb{N}$

1. $0 \le \varphi_j(x) \le 1$,
2. $\varphi_j(x) = 1$ for $x \in \tau B$,
3. $\operatorname{supp} \varphi_{j+1} \subset \{y \in B\ :\ \varphi_j(y) = 1\}$,
4. $\left\|\dfrac{|\sqrt{Q}\nabla\varphi_j|}{v^{\frac{1}{p}}}\right\|_\infty \le \dfrac{\tilde{C}}{r}$,

where $\tilde{C} > 0$ and $0 < \tau < 1$ are positive constants.

Now, we define the Q-weighted degenerate Sobolev space $QW^{1,p}(\Omega;\mu)$ and the solution of Eq. (16.1).

Definition 16.1 We denote by $QW^{1,p}(\Omega;\mu)$, $1 < p < \infty$ the completion of $Lip_{loc}(\Omega)$ with respect to the norm

$$\|u\|_{QW^{1,p}(\Omega;\mu)} = \left(\int_\Omega (|u|^p + (\nabla u\cdot Q\nabla u)^{p/2})d\mu\right)^{\frac{1}{p}}$$
$$= \left(\int_\Omega (|u|^p + |\sqrt{Q}\nabla u|^p)d\mu\right)^{\frac{1}{p}}.$$

Definition 16.2 We say that a function $u \in QW^{1,p}(\Omega;\mu)$ is a solution of (16.1) if for all $\phi \in Lip_0(\Omega)$

$$\int_\Omega |\sqrt{Q}(x)\nabla u|^{p-2}\nabla\phi(x)Q(x)\nabla u\, d\mu + \int_\Omega |f(x)|^{p-2}f(x)\phi(x)\, d\nu = 0\,.$$

The main tools to be exploited are a subrepresentation formula and also a Fefferman-Phong inequality. First of all, we remark that the segment property ensures us for each $x \in B_0 \Subset \Omega$ there is a collection of balls $\{B_j^x\}_{j=1}^{\infty}$ depending on x so that each of the following hold true for $j \in \mathbb{N}$:

- $B_j^x \subset B_0$,
- $r(B_j^x) \sim 2^{-j} r(B_0)$,
- $\rho(B_j^x, x) \leq C r(B_j^x)$ with C independent of x and j, and
- $B_j^x \cap B_{j+1}^x$ contains a metric ball S_j with $r(S_j) \sim r(B_j^x)$.

Theorem 1 in [10] allows us to state the following subrepresentation formula.

Theorem 16.1 *Let $p > 1$. For any ball B_0 with radius sufficiently small, and $u \in Lip_{loc}(\Omega)$ one has*

$$\left|u(x) - u_{B;\nu}\right| \leq C \sum_{j=0}^{\infty} r(B_j^x) \left(\frac{1}{\mu(B_j^x)} \int_{B_j^x} \left|\sqrt{Q}\ \nabla u\right|^p \ d\mu(y) \right)^{1/p} \tag{16.2}$$

a.e. $(d\nu)$ $x \in \Omega$.

Now we define Stummel—Kato classes and Morrey spaces.

Definition 16.3 Let $p > 1$, $B_0 = B(x_0, r)$ a ball and $\{B_j^x\}_{j=1}^{\infty}$ be a chain of balls as above. Set for $V \in L^1_{loc}(R^n)$

$$\eta_j(V; r) \equiv \sup_{x_0 \in R^n} \sup_{y \in B_0} \int_{B_0} |V(x)| \frac{r^p(B_j^x)}{\mu(B_j^x)} \chi_{B_j^x}(y) d\nu(x)$$

we say that V belongs to the space $\tilde{S}_p(R^n)$, if

$$\eta(V; r) \equiv \left(\sum_{j=1}^{\infty} (\eta_j(V; r))^{1/p} \right)^p < +\infty.$$

We say that V belongs to $S_p(R^n)$ if, in addition, we have $\lim_{r \to 0} \eta(V; r) = 0$. Moreover, we say that the $V \in S_p'(R^n)$ if there exists $\delta > 0$ such that

$$\int_0^{\delta} \frac{(\eta(V; t))^{\frac{1}{p-1}}}{t} dt < +\infty.$$

We say that V belongs to the Morrey space $M_\sigma(R^n)$, if there exist C and $\sigma > 0$ such that $\eta(V; r) \leq C r^\sigma$.

Subrepresentation formula (16.2) is useful in developing the following weighted Fefferman-Phong inequality where we exploit the following class.

Theorem 16.2 (Fefferman-Phong Inequality) *Let $1 < p < \infty$ and let V be any function in $\tilde{S}_p(R^n)$. Then, there is a constant $C > 0$ (independent of V) so that for* *any ball* B_0 *with* $0 < r = r(B_0) < \delta$ *and* $2B \Subset \Omega$, *one has*

$$\int_{B_0} |V(x)|\ |u(x) - u_{B_0;\nu}|^p\ d\nu \le C\eta(V;r)\int_{B_0} \left|\sqrt{Q}\nabla u\right|^p\ d\mu \tag{16.3}$$

for any $u \in Lip(B_0)$.

Proof From Theorem 16.1 and Hölder's inequality we get

$$\begin{aligned}
&\int_{B_0} |V(x)|\ |u(x) - u_{B_0;\nu}|^p\ d\nu(x)\\
&\qquad \le C\int_{B_0} |V(x)||u(x) - u_{B_0;\nu}|^{p-1}\\
&\qquad\quad \sum_{j=0}^{\infty} r(B_j(x))\left[\frac{1}{\mu(B_j(x))}\int_{B_j(x)} |\sqrt{Q(y)}\ \nabla u(y)|^p d\mu(y)\right]^{1/p} d\nu(x)\\
&= \sum_{j=0}^{\infty}\int_{B_0} |V(x)||u(x)-u_{B_0;\nu}|^{p-1}\left[\frac{r^p(B_j^x)}{\mu(B_j^x)}\int_{B_j^x} |\sqrt{Q(y)}\ \nabla u(y)|^p d\mu(y)\right]^{1/p} d\nu(x)\\
&\qquad \le C\sum_{j=0}^{\infty}\left[\int_{B_0} |V(x)||u(x) - u_{B_0;\nu}|^p d\nu(x)\right]^{1/p'}\\
&\qquad\quad \cdot\left[\int_{B_0} |V(x)|\frac{r^p(B_j^x)}{\mu(B_j^x)}\int_{B_j^x} |\sqrt{Q(y)}\ \nabla u(y)|^p d\mu(y)d\nu(x)\right]^{1/p}\\
&\qquad \le C\left[\int_{B_0} |V(x)||u(x) - u_{B_0;\nu}|^p d\nu(x)\right]^{1/p'}\\
&\qquad\quad \sum_{j=0}^{\infty}\left[\int_{B_0} |V(x)|\frac{r^p(B_j^x)}{\mu(B_j^x)}\int_{B_0} |\sqrt{Q(y)}\ \nabla u(y)|^p \chi_{B_j^x}(y)d\mu(y)d\nu(x)\right]^{1/p}\\
&\qquad \le C\left[\int_{B_0} |V(x)||u(x) - u_{B_0;\nu}|^p d\nu(x)\right]^{1/p'}\\
&\qquad\quad \sum_{j=0}^{\infty}\left[\int_{B_0} |\sqrt{Q(y)}\ \nabla u(y)|^p\int_{B_0} |V(x)|\frac{r^p(B_j^x)}{\mu(B_j^x)}\chi_{B_j^x}(y)d\nu(x)d\mu(y)\right]^{1/p}\\
&\qquad \le C\left[\int_{B_0} |V(x)||u(x) - u_{B_0;\nu}|^p d\nu(x)\right]^{1/p'}\\
&\qquad\qquad \left(\int_{B_0} |\sqrt{Q(y)}\ \nabla u(y)|^p d\mu(y)\right)^{1/p}\sum_{j=0}^{\infty}(\eta_j(V;r))^{1/p}
\end{aligned}$$

$$\leq C\left[\int_{B_0}|V(x)||u(x)-u_{B_0;\nu}|^p d\nu(x)\right]^{1/p'}$$
$$\left(\int_{B_0}|\sqrt{Q(y)}\,\nabla u(y)|^p d\mu(y)\right)^{1/p}(\eta(V;r))^{1/p},$$

from which (16.3) follows. □

Following the classical Moser iteration technique, as adapted by J. Serrin in [11] for the quasilinear case, and with the aid of the Fefferman-Phong inequality instead of Hölder plus Sobolev inequalities to estimate products of coefficients times test functions, we are able to prove the local boundedness of the solutions and the Harnack inequality for non negative solutions of Eq. (16.1) (see [7, 8]).

Theorem 16.3 (Local Boundedness) *Let $v \in QW^{1,p}(\Omega;\mu)$ be a solution of (16.1). Assume $|f|^{p-1} \in S'_p(\Omega)$. Then, there exists a positive constant C, independent of v, such that for any B_r for which $B_{4r} \subset \Omega$ we have*

$$\|v\|_{L^\infty_\nu(B_r)} \leq C\left(\frac{\nu(B_{2r})}{\mu(B_{2r})}\right)^{\frac{k}{p(k-1)}}\left\{\left(\frac{1}{\nu(B_{2r})}\int_{B_{2r}}|v|^p\,d\nu\right)^{\frac{1}{p}}+\eta\left(|f|^{p-1};2r\right)\right\}.$$

Theorem 16.4 (Harnack Inequality) *Let $v \in QW^{1,p}(\Omega;\mu)$ be a non negative solution of (16.1). Assume $|f|^{p-1} \in S'_p(\Omega,\nu)$. Then, there exists a positive constant C, independent of v, such that for any B_r for which $B_{4r} \subset \Omega$ we have*

$$\sup_{B_r} v \leq C\left\{\inf_{B_r} v+\eta\left(|f|^{p-1};2r\right)\right\}$$

As a simple consequence of Harnack inequality, we get some regularity results for weak solutions.

Theorem 16.5 *Let u be a weak solution of Eq. (16.1) in Ω and $|f|^{p-1} \in S'_p(\Omega)$. Then, u is continuous in Ω. Moreover if we assume $|f|^{p-1} \in M_\sigma(\Omega)$ then u is locally Hölder continuous in Ω.*

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References

1. Di Fazio, G., Zamboni, P.: Regularity for quasilinear degenerate elliptic equations. Math. Z. **253**(4), 787–803 (2006)
2. Di Fazio, G., Fanciullo, M.S., Zamboni, P.: Harnack inequality and regularity for degenerate quasilinear elliptic equations. Math. Z. **264**(3), 679–695 (2010)

3. Di Fazio, G., Fanciullo, M.S., Zamboni, P.: Regularity for a class of strongly degenerate quasilinear operators. J. Differ. Equ. **255**(11), 3920–3939 (2013)
4. Di Fazio, G., Fanciullo, M.S., Zamboni, P.: Harnack inequality for degenerate elliptic equations and sum operators. Commun. Pure Appl. Anal. **14**(6), 2363–2376 (2015)
5. Di Fazio, G., Fanciullo, M.S., Zamboni, P.: Harnack inequality and continuity of weak solutions for doubly degenerate elliptic equations. Math. Z. **292**, 1325–1336 (2019)
6. Di Fazio, G., Fanciullo, M.S., Zamboni, P.: Unique continuation for degenerate quasilinear equations and sum operators. Atti della Accademia Peloritana dei Pericolanti - Classe di Scienze Fisiche, Matematiche e Naturali **98**(S2), A5 (2020)
7. Di Fazio, G., Fanciullo, M.S., Zamboni, P.: Nonlinear elliptic equations related to weighted sum operators. Nonlinear Anal. **194**, 111570 (2020)
8. Di Fazio, G., Fanciullo, M.S., Zamboni, P.: Degenerate elliptic equations and Sum operators. In: New Trends in Analysis and Geometry. Cambridge Scholars Publishing, Newcastle upon Tyne, pp. 45–80 (2020)
9. Di Fazio, G., Fanciullo, M.S., Monticelli, D.D., Rodney, S., Zamboni, P.: Matrix weights and regularity for degenerate elliptic equations. Nonlinear Anal. **237**, 113363 (2023)
10. Franchi, B., Perez, C., Wheeden, R.L.: A sum operator with applications to self–improving properties of poincaré inequalities in metric spaces. J. Fourier Anal. Appl. **9**(5), 511–540 (2003)
11. Serrin, J.: Local behavior of solutions of quasi-linear equations. Acta Math. **111**, 247–302 (1964)

Chapter 17
Stability Analysis of Differential Equations Using Mohand Integral Transform

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Abstract The Mohand integral transform is used in this study to examine the stability in Ulam-Hyers sense of linear differential equations. Finally, the Mohand integral transform is used to derive the stability results. These results are shown in figures depicting the behavior of an RLC series circuit.

17.1 Introduction

The well-known Ulam-Hyers stability problem originated in the period of 1940–1941, initially proposed by Ulam [15] and Hyers [3]. Rassias [10] popularized the concept of stability. In recent decades, numerous mathematicians have been dedicated to the exploration of stability theory in the Ulam-Hyers sense for covering various functional and differential equations [9, 11, 12].

Mohand and Mahgoub were the first to suggest the Mohand transform, which gives us the classical Laplace transform is the foundation of the Mohand transform method, however it is enhanced and modified in [5]. In [7], the authors examined the dynamic properties of fractional derivatives of both Riemann-Liouville and Caputo types and derived results for their fractional derivatives. The results obtained through Mohand transform is very close to the results obtained by Laplace transform [1]. In order to convey that Mohand transform can be replaced with Laplace transform to solve differential equations, we have considered Mohand transform in this study. A lot of authors have contributed to stability in the Ulam-Hyers sense by

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using integral transform methods on different kinds of differential equations. These include the Laplace transform [2], Mahgoub transform [4], Aboodh transform [13] and general transform [8].

Motivated by existing literature, this study employs the Mohand integral transform to investigate the stability in the Ulam-Hyers sense of the second-order linear differential equations as follows:

$$h''(s) + ph'(s) + qh(s) = 0 \tag{17.1}$$

and

$$h''(s) + ph'(s) + qh(s) = \psi(s), \tag{17.2}$$

where p and q are constants, and $h(s)$ is a twice continuous differentiable function. Furthermore, we provide an application that examples of the derived abstract theory.

17.2 Preliminary Concepts Projected Method

This section recalls basic concepts in the field of Mohand integral transform, as described in the literature. In this paper, the notation $\mathcal{W}$ is employed to denote either the real field $\mathcal{R}$ or the complex field $\mathcal{C}$. A function $h(s)$ is considered to be of exponential order if there exist constants $\mathcal{A}, \mathcal{B} \in \mathcal{R}$ such that $|h(s)| \leq \mathcal{A}e^{\mathcal{B}s}$ for all $s \geq 0$.

Definition 17.1 ([14]) The Mohand integral transform of the function $h(s)$ is given by the formula:

$$\mathcal{M}\{h(s)\} = \omega^2 \int_0^\infty h(s)e^{-\omega s} ds = \mathcal{H}(\omega),$$

where $\mathcal{M}$ is the Mohand transform operator. The inverse of the Mohand transform operator $\mathcal{H}(\omega)$ is denoted by $h(s)$ can be obtained as $\mathcal{M}^{-1}\{\mathcal{H}(\omega)\} = h(s)$, where $\mathcal{M}^{-1}$ is the inverse Mohand transform operator.

Definition 17.2 ([1]) If Mohand transform of functions $h(s)$ and $g(s)$ are $\mathcal{H}(\omega)$ and $\mathcal{G}(\omega)$ respectively, then Mohand transform of their convolution $h(s) * g(s)$ is given by

$$h(s) * g(s) = \mathcal{M}\{h(s)\} * \mathcal{M}\{g(s)\} = \frac{1}{\omega^2}\mathcal{H}(\omega)\mathcal{G}(\omega) = \int_0^\infty h(u)g(s-u)du.$$

Lemma 17.1 ([1, 7]) *If* $\mathcal{M}\{h(s)\} = \mathcal{H}(\omega)$, *then*

- $\mathcal{M}\{h'(s)\} = \omega\mathcal{H}(\omega) - \omega^2 h(0)$,
- $\mathcal{M}\{h''(s)\} = \omega^2\mathcal{H}(\omega) - \omega^3 h(0) - \omega^2 h'(0)$.
- $\mathcal{M}\{h^{(n)}(s)\} = \omega^n\mathcal{H}(\omega) - \omega^{n+1} h(0) - \omega^n h'(0) - \cdots - \omega^2 h^{n-1}(0)$.

17.3 Stability Analysis of the Projected Problem

To discuss the stability analysis of the projected problem. The definition that follows is taken from [6] and modified for the proposed method.

Definition 17.3 ([6]) Linear problem (17.1) has Ulam–Hyers stability if there exists a constant $\mathcal{Z} > 0$ such that for a given $\epsilon > 0$ and for each function $h(s)$ satisfying the following inequality

$$\left|h''(s) + ph'(s) + qh(s)\right| \leq \epsilon, \ \forall\, s \geq 0, \tag{17.3}$$

then there exists a solution $\rho(s)$ of the (17.1), such that $|h(s) - \rho(s)| \leq \mathcal{Z}\epsilon, \forall\, s \geq 0$, where $\mathcal{Z}$ represents a constant for Ulam-Hyers stability.

Definition 17.4 ([6]) Linear problem (17.2) has Ulam–Hyers stability if there exists a constant $\mathcal{Z} > 0$ such that for a given $\epsilon > 0$ and for each function $h(s)$ satisfying the following inequality

$$\left|h''(s) + ph'(s) + qh(s) - \psi(s)\right| \leq \epsilon, \forall\, s \geq 0. \tag{17.4}$$

Then there exists a solution $\rho(s)$ of the (17.2), such that $|h(s) - \rho(s)| \leq \mathcal{Z}\epsilon, \forall\, s \geq 0$, where $\mathcal{Z}$ represents a constant for Ulam-Hyers stability.

Theorem 17.1 *Let p and q be given constants. If a function $h(s)$ satisfying the inequality (17.3), then there exists a solution $\rho(s)$ of the inequality (17.1) with*

$$|h(s) - \rho(s)| \leq \mathcal{Z}\epsilon, \forall\, s \geq 0.$$

Proof Let us assume that $h(s)$ is a twice continuously differentiable function that satisfies the inequality (17.3). A function $d(s)$ should be defined as follows:

$$d(s) := h''(s) + ph'(s) + qh(s), \ \forall\, s \geq 0. \tag{17.5}$$

Applying the Mohand integral transform to (17.5), we get

$$\mathcal{D}(\omega) = \Big(\omega^2\mathcal{H}(\omega) - \omega^3 h(0) - \omega^2 h'(0)\Big) + p\Big(\omega\mathcal{H}(\omega) - \omega^2 h(0)\Big) + q\mathcal{H}(\omega).$$

Now,

$$\mathcal{H}(\omega) = \frac{\mathcal{D}(\omega) + \omega^3 h(0) + \omega^2 h'(0) + p\omega^2 h(0)}{\omega^2 + p\omega + q}.$$

Since for every p and q are constants, there exists a constant $\lambda, \mu \in \mathcal{W}$ such that $\lambda + \mu = -p$ and $\lambda\mu = q$, where $\lambda \neq \mu$ then

$$\omega^2 + p\omega + q = (\omega - \lambda)(\omega - \mu).$$

Thus,

$$\mathcal{H}(\omega) = \frac{\mathcal{D}(\omega) + \omega^3 h(0) + \omega^2 h'(0) + p\omega^2 h(0)}{(\omega - \lambda)(\omega - \mu)}. \tag{17.6}$$

Then $\rho(0) = h(0)$ and $\rho'(0) = h'(0)$. The Mohand transform of $\rho(s)$ yields as follows:

$$\mathcal{P}(\omega) = \frac{\omega^3 h(0) + \omega^2 h'(0) + p\omega^2 h(0)}{(\omega - \lambda)(\omega - \mu)}. \tag{17.7}$$

Hence, we get

$$\begin{aligned} \mathcal{M}\Big\{\rho''(s) + p\rho'(s) + q\rho(s)\Big\} &= \Big(\omega^2\mathcal{P}(\omega) - \omega^3\rho(0) - \omega^2\rho'(0)\Big) \\ &\quad + p\Big(\omega\mathcal{P}(\omega) - \omega^2\rho(0)\Big) + q\mathcal{P}(\omega), \\ \mathcal{P}(\omega) &= \frac{\omega^3\rho(0) + \omega^2\rho'(0) + p\omega^2\rho(0)}{(\omega - \lambda)(\omega - \mu)}. \end{aligned}$$

As the operator $\mathcal{M}$ is one-to-one,

$$\rho''(s) + p\rho'(s) + q\rho(s) = 0.$$

Hence, $\rho(s)$ is a solution of (17.1). Further, it follows from (17.6) and (17.7), we get

$$\mathcal{H}(\omega) - \mathcal{P}(\omega) = \frac{\mathcal{D}(\omega)}{(\omega - \lambda)(\omega - \mu)} = \mathcal{M}\Big\{d(s) * \ell(s)\Big\},$$

where

$$\ell(s) = \mathcal{M}^{-1}\left\{\frac{1}{(\omega - \lambda)(\omega - \mu)}\right\}.$$

These equalities show that

$$h(s) - \rho(s) = d(s) * \ell(s).$$

By taking the modulus on both sides, we obtain

$$\Big|h(s) - \rho(s)\Big| = \Big|d(s) * \ell(s)\Big| \leq \Big|\int_0^s d(s)\ell(s-u)du\Big|.$$

Given inequality (17.3), we get $|d(s)| \leq \epsilon$,

$$\begin{aligned}\Big|h(s) - \rho(s)\Big| &\leq \int_0^s \Big|d(s)\Big|\Big|\ell(s-u)du\Big| \\ &\leq \mathcal{Z}\epsilon,\ \forall\, s \geq 0,\ \mathcal{Z} = \Big|\int_0^s \ell(s-u)du\Big|.\end{aligned}$$

Therefore, linear problem (17.1) has Ulam–Hyers stability. □

Theorem 17.2 *Let p and q be given constants. If a function $h(s)$ satisfies the inequality (17.4), then there exists a solution $\rho(s)$ of inequality (17.2) with*

$$|h(s) - \rho(s)| \leq \mathcal{Z}\epsilon, \forall\, s \geq 0.$$

Proof Let us assume that $h(s)$ is a twice continuously differentiable function that satisfies inequality (17.4). A function $d(s)$ should be defined as follows:

$$d(s) := h''(s) + ph'(s) + qh(s) - \psi(s),\ \forall\, s \geq 0. \tag{17.8}$$

Applying the Mohand integral transform to (17.8), we get

$$\mathcal{H}(\omega) = \frac{\mathcal{D}(\omega) + \omega^3 h(0) + \omega^2 h'(0) + p\omega^2 h(0) + \mathcal{M}\{\psi(s)\}}{\omega^2 + p\omega + q}.$$

Since for every p and q are constants, there exists a constant $\lambda, \mu \in \mathcal{W}$ such that $\lambda + \mu = -p$ and $\lambda\mu = q$, where $\lambda \neq \mu$ then

$$\omega^2 + p\omega + q = (\omega - \lambda)(\omega - \mu).$$

Thus,

$$\mathcal{H}(\omega) = \frac{\mathcal{D}(\omega) + \omega^3 h(0) + \omega^2 h'(0) + p\omega^2 h(0) + \mathcal{M}\{\psi(s)\}}{(\omega - \lambda)(\omega - \mu)}. \tag{17.9}$$

Then, $\rho(0) = h(0)$ and $\rho'(0) = h'(0)$. The Mohand integral transform of $\rho(s)$ yields as follows:

$$\mathcal{P}(\omega) = \frac{\omega^3 h(0) + \omega^2 h'(0) + p\omega^2 h(0) + \mathcal{M}\{\psi(s)\}}{(\omega - \lambda)(\omega - \mu)}. \tag{17.10}$$

Hence, we get

$$\mathcal{M}\Big\{\rho''(s) + p\rho'(s) + q\rho(s) - \psi(s)\Big\} = \Big(\omega^2\mathcal{H}(\omega) - \omega^3 h(0) - \omega^2 h'(0)\Big)$$
$$+ p\Big(\omega\mathcal{H}(\omega) - \omega^2 h(0)\Big) + q\mathcal{H}(\omega) + \mathcal{M}\{\psi(s)\},$$
$$\mathcal{P}(\omega) = \frac{\omega^3 \rho(0) + \omega^2 \rho'(0) + p\omega^2 \rho(0)) + \mathcal{M}\{\psi(s)\}}{(\omega - \lambda)(\omega - \mu)}.$$

As the operator $\mathcal{M}$ is one-to-one,

$$\rho''(s) + p\rho'(s) + q\rho(s) - \mathcal{M}\{\psi(s)\} = 0.$$

Hence $\rho(s)$ is a solution of (17.2). Further, it follows from (17.9) and (17.10), we get

$$\mathcal{H}(\omega) - \mathcal{P}(\omega) = \frac{\mathcal{D}(\omega)}{(\omega - \lambda)(\omega - \mu)} = \mathcal{M}\Big\{d(s) * \ell(s)\Big\},$$

where

$$\ell(s) = \mathcal{M}^{-1}\left\{\frac{1}{(\omega - \lambda)(\omega - \mu)}\right\}.$$

These equalities show that

$$h(s) - \rho(s) = d(s) * \ell(s).$$

Taking the modulus from both sides gives us

$$\Big|h(s) - \rho(s)\Big| = \Big|d(s) * \ell(s)\Big| \leq \Big|\int_0^s d(s) * \ell(s - u)du\Big|.$$

Given inequality (17.4), we get $|d(s)| \leq \epsilon$,

$$\Big|h(s) - \rho(s)\Big| \leq \int_0^s \Big|d(s)\Big|\Big|d(s) * \ell(s-u)du\Big| \leq \mathcal{Z}\epsilon,\ \forall\, s \geq 0,$$

$$\mathcal{Z} = \Big|\int_0^s \ell(s-u)du\Big|.$$

Therefore, linear problem (17.2) is Ulam-Hyers stable. □

17.4 Application of the Projected Method

Here, we analyse Ulam–Hyers stability of the RLC series circuit system with the projected method.

The RLC series circuit shown in Fig. 17.1, can be described in the equation as follows:

$$L\frac{d^2 I}{ds^2} + R\frac{dI}{ds} + \frac{I}{C} = V_I(s),$$

with initial condition $I(0) = I_0$, $I'(0) = I_0'$. The parameters are taken as armature current I in ampere (A), inductance L in henry (H), resistance R in ohm (Ω), capacitor C in farad (F) and voltage V_I in volt.

Example 17.1 Find the current $I(s)$ for the RLC series circuit with zero initial current using the Mohand transform at time s. The total voltage force is 50 to a series in which the inductance is $0.15\mu H$, the resistance is 12Ω and a capacitor is $100pF$.

Let us consider the RLC series circuit equation in the form:

$$0.15I''(s) + 12I' + 0.01I(s) = 50. \tag{17.11}$$

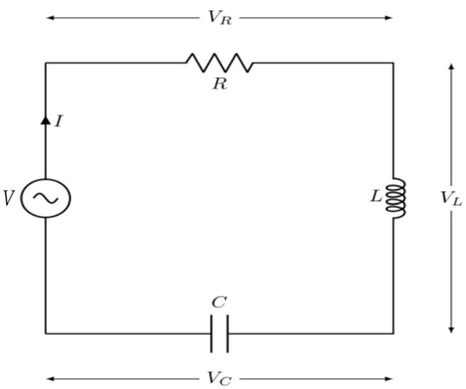

Fig. 17.1 Diagram for RLC series circuit

Dividing the Eq. (17.11) by 0.15, we get

$$I^{''}(s) + 80I^{'}(s) + 0.0667I(s) = 333.33,$$

we put the initial current as $I(0) = 0,\ I'(0) = 0$. Define the function $d(s) = I^{''}(s)+80I^{'}(s)+0.0667I(s)-333.33$, if twice continuously differentiable function $I(s)$ satisfies

$$\left|I^{''}(s) + 80I^{'}(s) + 0.0667I(s) - 333.33\right| \leq \epsilon, \forall\, s \geq 0.$$

Applying the Mohand integral transform to $d(s)$ function, we get

$$\mathcal{I}(\omega) = \frac{\mathcal{D}(\omega) + \mathcal{M}\{333.33\}}{(\omega - 80)(\omega - 0.0667)}, \tag{17.12}$$

$$\mathcal{I}_0(\omega) = \frac{\mathcal{M}\{333.33\}}{(\omega - 80)(\omega - 0.0667)}. \tag{17.13}$$

By using (17.12) into (17.13), we obtain

$$\begin{aligned}\mathcal{I}(\omega) - \mathcal{I}_0(\omega) &= \frac{\mathcal{D}(\omega)}{(\omega - 80)(\omega - 0.0667)}\\ &= \mathcal{M}\Big\{d(s) * \ell(s)\Big\},\ \ \ell(s) = e^{-(80+0.0667)s}.\end{aligned}$$

These equalities show that

$$I(s) - I_0(s) = d(s) * e^{-(80.0667)s}.$$

Taking the modulus from both sides gives us

$$\left|I(s) - I_0(s)\right| \leq \textstyle\int_0^s \left|d(s)\right| \left| \textstyle\int_0^s e^{-(80.0667)(s-u)} du\right| \leq \mathcal{Z}\epsilon,\ \ \forall\, s \geq 0,$$

where,

$$\mathcal{Z} = \left| \int_0^s e^{-(80.0667)(s-u)} du\right| = \frac{1}{80.0667}(1 - e^{-80.0667s}) = \frac{1}{80.0667}.$$

Therefore, the RLC series circuit system (17.11) has Ulam-Hyers stability.

Example 17.2 A series RLC circuit containing a resistance of 10Ω, an inductance of 0.15 μH, and a capacitor of 500 pF are connected in series across a total voltage force of 36 supply. Find the current $I(s)$ for the RLC series circuit with zero initial current using the Mohand transform at time s.

Let us choose the RLC series circuit equation form:

$$0.15I^{''}(s) + 10I^{'}(s) + 0.002I(s) = 36. \tag{17.14}$$

Dividing Eq. (17.14) by 0.15, we get

$$I^{''}(s) + 66I^{'}(s) + 0.0133I(s) = 240,$$

we put the initial current as $I(0) = 0$ and $I'(0) = 0$. Consider $d(s) = I^{''}(s) + 66I(s) + 0.0133I(s) - 240$, if a twice continuously differentiable function $I(s)$ satisfies

$$\left| I^{''}(s) + 66I(s) + 0.0133I(s) - 240 \right| \leq \epsilon, \forall\, s \geq 0,$$

$$\mathcal{I}(\omega) = \frac{\mathcal{D}(\omega) + \mathcal{M}\{240\}}{(\omega - 66)(\omega - 0.0133)}, \tag{17.15}$$

$$\mathcal{I}_0(\omega) = \frac{\mathcal{M}\{240\}}{(\omega - 66)(\omega - 0.0133)}. \tag{17.16}$$

By using (17.15) into (17.16), we obtain

$$\begin{aligned} \mathcal{I}(\omega) - \mathcal{I}_0(\omega) &= \frac{\mathcal{D}(\omega)}{(\omega - 66)(\omega - 0.0133)} \\ &= \mathcal{M}\Big\{d(s) * \ell(s)\Big\},\ \ell(s) = \mathcal{M}^{-1}\Big\{e^{-(66+0.0133)s}\Big\}. \end{aligned}$$

These equalities show that

$$I(s) - I_0(s) = d(s) * e^{-(66.0133)s}.$$

Taking the modulus from both sides gives us

$$\left| I(s) - I_0(s) \right| \leq \int_0^s \left| d(s) \right| \left| \int_0^s e^{-(66.0133)(s-u)} du \right| \leq \mathcal{Z}\epsilon,\ \forall\, s \geq 0,$$

where

$$\mathcal{Z} = \left| \int_0^s e^{-(66.0133)(s-u)} du \right| = \frac{1}{66.0133}(1 - e^{-66.0133s}) = \frac{1}{66.0133}.$$

Therefore, the RLC series circuit system (17.14) has Ulam-Hyers stability.

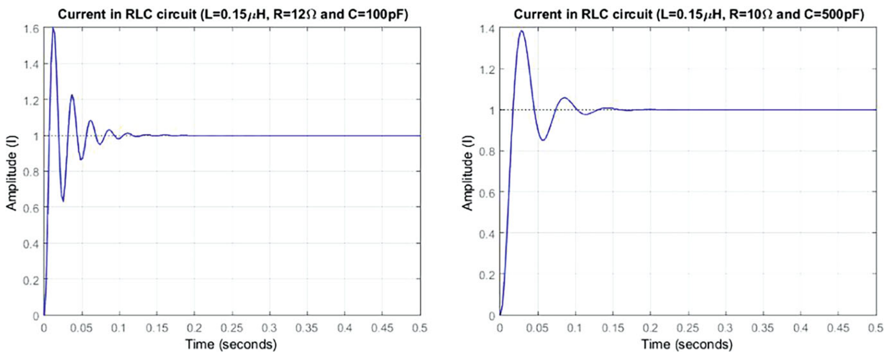

Fig. 17.2 Step responses against time for the systems given by (17.11) and (17.14)

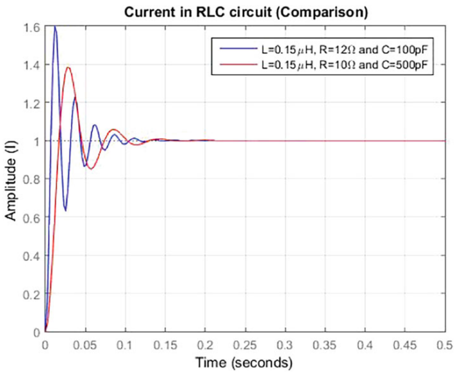

Fig. 17.3 Comparison of step response and against time for the systems given by (17.11) and (17.14)

In Fig. 17.2, the results show that the system described by Eq. (17.11) has a settling time of 0.0898 s and the system governed by Eq. (17.14) has a settling time of 0.0774 s. The two systems are compared in Fig. 17.3. The analysis indicates that both systems are stable.

17.5 Conclusion

In this investigation, we derived results regarding the stability in the Ulam-Hyers sense of the linear differential equations by employing the Mohand integral transform. Furthermore, we successfully established an application for stability results based on this consideration. The Mohand integral transform for Ulam-Hyers stability has been shown to work in the real world by looking at an electrical RLC series circuit system.

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Competing Interests The authors declare that they have no competing interests.

Ethics Approval This article does not contain any studies with human participants or animals performed by any authors.

References

1. Aggarwal, S., Chaudhary, R.: A comparative study of Mohand and Laplace transforms. J. Emerg. Technol. Innov. Res. **6**, 230–240 (2019)
2. Alqifiary, Q.H., Jung, S.M.: Laplace transform and generalized Hyers-Ulam stability of linear differential equations. Electron. J. Differ. Equ. **2014**, 1–11 (2014)
3. Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. **27**, 222–224 (1941)
4. Jung, S.M., Ponmana Selvan, A., Ramdoss, M.: Mahgoub transform and Hyers-Ulam stability of first-order linear differential equations. J. Math. Inequal. **15**, 1201–1218 (2021)
5. Mohand, M., Mahgoub, A.: The new integral transform "Mohand Transform". Adv. Theor. Appl. Math. **12**, 113–120 (2017)
6. Murali, R., Ponmana Selvan, A., Park, C., Lee, J.R.: Aboodh transform and the stability of second order linear differential equations. Adv. Differ. Equ. **2021**, 296 (2021)
7. Patra, A., Baliarsingh, P., Dutta, H.: Solution to fractional evolution equation using Mohand transform. Math. Comput. Simul. **200**, 557–570 (2022)
8. Pinelas, S., Selvam, A., Sabarinathan, S.: Ulam-Hyers stability of linear differential equation with general transform. Symmetry **15**, 1–12 (2023)
9. Ramdoss, M., Ponamana Selvan, A., Park, C.: Ulam stability of linear differential equations using Fourier transform. AIMS Math. **5**, 766–780 (2020)
10. Rassias, T.M.: On the stability of the linear mappings in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
11. Rassias, J.M., Murali, R., Ponmana Selvan, A.: Mittag-Leffler-Hyers-Ulam stability of linear differential equations using Fourier transforms. J. Comput. Anal. Appl. **29**, 68–85 (2021)
12. Selvam, A., Sabarinathan, S., Noeiaghdam, S., Govindan, V.: Fractional Fourier transform and Ulam stability of fractional differential equation with fractional Caputo-type derivative. J. Funct. Spaces **2022**, 1–5, 777566 (2022)
13. Selvam, A., Sabarinathan, S., Pinelas, S.: The Aboodh transform techniques to Ulam type stability of linear delay differential equation. Int. J. Appl. Comput. Math. **9**, 115 (2023)
14. Shah, R., Farooq, U., Khan, H., Baleanu, D., Kumam, P., Arif, M.: Fractional view analysis of third order Kortewege-De Vries equations, using a new analytical technique. Front. Phys. **7**, 244 (2020)
15. Ulam, S.M.: Problem in Modern Mathematics. Wiley, New York (1964)

Part IV
Modeling and Applications

Chapter 18
Mathematical Issues of Difference Schemes for Atmospheric Boundary Layer Equations

Dinara Tamabay, Bakytzhan Zhumagulov, and Almas Temirbekov

Abstract This research investigates the dispersion of pollutants in the air of Ust-Kamenogorsk city. Employing a model of the atmospheric boundary layer, the study focuses on simulating and forecasting the dissemination of detrimental substances in the atmosphere. Using the finite difference method and the splitting scheme for physical processes, the difference schemes for the model under consideration are presented. Also in this paper, the approximation, stability and convergence of the constructed difference scheme for calculation are presented. Numerical experiments and visualization of the results are carried out.

18.1 Introduction

The quality of the urban atmospheric air is compromised due to emissions from significant sources such as large thermal power facilities, substantial vehicular traffic, activities in the private sector, and industrial enterprises; this adverse effect is exacerbated by unfavorable atmospheric conditions and the city's geographical location [1].

A model of the atmospheric boundary layer is used to simulate the spread of pollution in the atmospheric air. The model is based on multidimensional non-stationary tasks. The model takes into account the continuous distribution of air pressure under the influence of heat transfer processes. The fundamental principles of weather and climate are shaped by the interplay of atmospheric motions and thermodynamic processes. These principles serve as the foundation for addressing various practical challenges, with a primary focus on environmental issues. The

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relevance of the problem is confirmed by publications devoted to this topic. The works of Marchuk G.I. [2], Penenko V.V. [3], Aloyan A.E. [4], Arguchintsev V.K. [5] are devoted to the mathematical modeling of the process of atmospheric air pollution.

To study atmospheric pollution, Olsen H.R., et al. [6] used Gaussian models considering the propagation of gases and particles from point sources. In the Republic of Kazakhstan, Zakarin E.A. [7, 8], Danaev N.T. et al. [9–11], Malgazhdarov Y.A. [12] are engaged in the creation of geographic information systems in the field of ecology. The works of Smagulov S.S., Danaev N.T., Zhumagulov B.T., Temirbekov N.M. [13, 14] are devoted to the mathematical substantiation of the numerical solution of problems, that are on the basis of considering system, as well as the works of Temirbekov A.N. et al. [15–18], and many others.

The purpose of this study is to simulate atmospheric air pollution in the city of Ust-Kamenogorsk using a model of the boundary layer of the atmosphere.

18.2 Formulation of the Problem

A model of the atmospheric boundary layer of the following type is considered, equations of motion are as follows:

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial u\omega}{\partial z}=-\frac{\partial p}{\partial x}+lv+\frac{\partial}{\partial x}(\mu_x\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(\mu_y\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(\nu\frac{\partial u}{\partial z}), \tag{18.1}$$

$$\frac{\partial v}{\partial t}+\frac{\partial vu}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial v\omega}{\partial z}=-\frac{\partial p}{\partial y}-lu+\frac{\partial}{\partial x}(\mu_x\frac{\partial v}{\partial x})+\frac{\partial}{\partial y}(\mu_y\frac{\partial v}{\partial y})+\frac{\partial}{\partial z}(\nu\frac{\partial v}{\partial z}), \tag{18.2}$$

$$\frac{\partial \omega}{\partial t}+\frac{\partial \omega u}{\partial x}+\frac{\partial v\omega}{\partial y}+\frac{\partial \omega^2}{\partial z}=-\frac{\partial p}{\partial z}+\lambda\theta+\frac{\partial}{\partial x}(\mu_x\frac{\partial \omega}{\partial x})+\frac{\partial}{\partial y}(\mu_y\frac{\partial \omega}{\partial y})+\frac{\partial}{\partial z}(\nu\frac{\partial \omega}{\partial z}), \tag{18.3}$$

the continuity equation is expressed as:

$$div\,\overrightarrow{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial z}=0, \tag{18.4}$$

heat inflow equations are considered as:

$$\frac{\partial \theta}{\partial t} + \frac{\partial \theta u}{\partial x} + \frac{\partial v\theta}{\partial y} + \frac{\partial \omega\theta}{\partial z} + u(S\frac{\partial \delta}{\partial x}\theta_x) + v(S\frac{\partial \delta}{\partial y} + \theta_x) = \frac{\partial}{\partial x}(\mu_x \frac{\partial \theta}{\partial x}) + \frac{\partial}{\partial y}(\mu_y \frac{\partial \theta}{\partial y}) + \frac{\partial}{\partial z}(\nu \frac{\partial \theta}{\partial z}). \tag{18.5}$$

To model the transfer and transformation of impurities, we use the following equations:

$$\frac{\partial \varphi_q}{\partial t} + \frac{\partial \varphi_q u}{\partial x} + \frac{\partial v\varphi_q}{\partial y} + \frac{\partial \omega\varphi_q}{\partial z} = \frac{\partial}{\partial x}(\mu_x \frac{\partial \varphi_q}{\partial x}) + \frac{\partial}{\partial y}(\mu_y \frac{\partial \varphi_q}{\partial y}) + \frac{\partial}{\partial z}(\nu \frac{\partial \varphi_q}{\partial z}) + \alpha_q\varphi_q + \beta_q + f_q, \sum_q \varphi_q = 1, \tag{18.6}$$

where $\overrightarrow{V}$—vector representing the velocity of the wind with its respective components u, v, ω; p—pressure indicator, λ—parameter related to convection, S—parameter of stratification, μ_x, μ_y—coefficients of horizontal turbulence for both motion and heat; ν—coefficient governing turbulent exchange vertically for both motion and heat, θ—the ambient potential temperature, l—parameter of Coriolis, φ_q—the ratio of concentrations of pollutants in the impurity, f_q—explains the origins of substances at the roughness level, α_q, β_q—coefficients resulting from atmospheric impurity transformation equations, where the index denotes the chemical formula of the substance.

The initial and boundary conditions applied for solving the system of Eqs. (18.1)–(18.6) are as follows:

$$\overrightarrow{V} = \overrightarrow{V}^0(x, y, z), \theta = \theta^0(x, y, z), \varphi_q = \varphi_q^0(x, y, z), t = 0,$$

$$u = u_1(y, z, t), v = v_1(y, z, t), \omega = 0, \frac{\partial \theta}{\partial x} = 0,$$
$$\varphi_q = \varphi_q^0, x = 0, 0 \le y \le Y, h \le z \le H,$$

$$\frac{\partial u}{\partial x} = 0, v = 0, \omega = 0, \frac{\partial \theta}{\partial x} = 0, \frac{\partial \varphi_q}{\partial x} = 0, x = X, 0 \le y \le Y, h \le z \le H,$$

$$u = u_2(x, z, t), v = v_2(x, z, t), \omega = 0, \frac{\partial \theta}{\partial y} = 0,$$
$$\varphi_q = \varphi_q^0, y = 0, 0 \le x \le X, h \le z \le H,$$

$$u = 0, \frac{\partial v}{\partial x} = 0, \omega = 0, \frac{\partial \theta}{\partial y} = 0, \frac{\partial \varphi_q}{\partial y} = 0, y = Y, 0 \le x \le X, h \le z \le H, \tag{18.7}$$

$$u = 0, v = 0, \omega = 0, p = 0, \varphi_q = 0, z = H, 0 \le x \le X, 0 \le y \le Y,$$

$$\omega = 0, h\frac{\partial u}{\partial z} = a_u u, h\frac{\partial v}{\partial z} = a_u v, h\frac{\partial \theta}{\partial z} = a_\theta(\theta - \theta_0), z = h, 0 \le x \le X, 0 \le y \le Y.$$

Here, H is the conditional height; X, Y—the dimensions of the horizontal boundaries of the calculated area, $\varphi_{q,0}, \varphi_{q,h}$ are the proportion of concentrations of matter q at roughness degree in the surface layer, θ_q is temperature, $a_u = \frac{\psi_u(\zeta_h)}{\eta_u(\zeta_h,\zeta_0)}$ is the parameter induced by the friction between air flows and the underlying surface, $a_\theta = \frac{\psi_\theta(\zeta_h)}{\eta_\theta(\zeta_h,\zeta_0)}$ is the parameter governing turbulent heat transfer, h is height of surface layer, ζ_h, ζ_0 are the parameters of dimensionless height, ψ_u, ψ_θ are Businger interpolation functions. The boundary conditions of the type $u = u_1(y, z, t), v = v_1(y, z, t)$ are set based on meteorological conditions. When setting boundary conditions for θ, φ_q in the surface layer $z = h$, the interactions of impurities with the underlying surface are taken into account.

In order to solve the above problem, we will consider the uniform grid in region $\Omega = 0 \le x \le l_1, 0 \le y \le l_2, 0 \le z \le l_3$, where $h_1 = l_1/N_1, h_2 = l_2/N_2, h_3 = l_3/N_3$:

$$\Omega_h = \{(x_i, y_j, z_k) = (ih_1, jh_2, kh_3), i = 0, \ldots, N_1, j = 0, \ldots, N_2, k = 0, \ldots, N_3\},$$

$$\Omega_{x,h} = \{(x_{i-1/2}, y_j, z_k) = ((i - 1/2)h_1, jh_2, kh_3), i = 1, \ldots, N_1, j = 0, \ldots, N_2, k = 0, \ldots, N_3\},$$

$$\Omega_{y,h} = \{(x_i, y_{j-1/2}, z_k) = (ih_1, (j - 1/2)h_2, kh_3), i = 0, \ldots, N_1, j = 1, \ldots, N_2, k = 0, \ldots, N_3\},$$

$$\Omega_{z,h} = \{(x_i, y_j, z_{k-1/2}) = (ih_1, jh_2, (k - 1/2)h_3), i = 0, \ldots, N_1, j = 0, \ldots, N_2, k = 1, \ldots, N_3\}.$$

Components of the wind speed u, v, ω are determined in the grid nodes $\Omega_{x,h}$, $\Omega_{y,h}$, $\Omega_{z,h}$, respectively. Temperature θ is determined in the grid nodes $\Omega_{z,h}$, and pressure p, the ratio of concentrations of harmful substances φ_q are determined in the grid nodes Ω_h.

18.3 Numerical Implementation Algorithm

In order to find $\overrightarrow{V}_h^{n+1}$, θ_h^n, $\varphi_{q,h}^n$ at a time t^{n+1}, a splitting scheme is used, which consists of the following stages:

1. Determination of $\overrightarrow{V}_h^{n+1/2}$ is realized as:

$$\frac{\overrightarrow{V}_h^{n+1/2} - \overrightarrow{V}_h^n}{\tau} = -L_h \overrightarrow{V}_h^n + \Lambda_h \overrightarrow{V}_h^n + \overrightarrow{G}_h^n, \tag{18.8}$$

   where $\overrightarrow{G}_h = (lv_h^n, -lu_h^n, \lambda\theta_h^n)$.
2. According to the found $\overrightarrow{V}_h^{n+1/2}$, taking into account that $div_h \overrightarrow{V}^{n+1} = 0$, the following equation for p is obtained:

$$\Lambda_h p_h^{n+1} = \frac{div \overrightarrow{V}_h^{n+1/2}}{\tau}. \tag{18.9}$$

3. To determine the speed $\overrightarrow{V}_h^{n+1}$ on the time layer t^{n+1}, we have

$$\frac{\overrightarrow{V}_h^{n+1} - \overrightarrow{V}_h^{n+1/2}}{\tau} = -\nabla_h p_h^{n+1}. \tag{18.10}$$

4. According to $\overrightarrow{V}_h^{n+1}$, the temperature transfer and diffusion are computed using the difference scheme presented below:

$$\frac{\theta_h^{n+1} - \theta_h^n}{\tau} = -L_h\theta_h^n + \Lambda_h\theta_h^n. \tag{18.11}$$

5. Conversion of portions of harmful substance concentrations into impurities through convection and turbulent exchange is determined as presented below:

$$\frac{\varphi_{q,h}^{n+1/2} - \varphi_{q,h}^n}{\tau} = L_h\varphi_{q,h}^n + \Lambda_h\varphi_{q,h}^n. \tag{18.12}$$

6. The alteration of harmful substance concentrations into impurities and the impact of external sources is determined as:

$$\frac{\varphi_{q,h}^{n+1} - \varphi_{q,h}^{n+1/2}}{\tau} = \alpha_q\varphi_{q,h}^{n+1} + \beta_q + f_q. \tag{18.13}$$

For example, for the fraction of the concentration of a substance HSO_3, we have:

$$\frac{\varphi_{HSO_3}^{n+1} - \varphi_{HSO_3}^{n+1/2}}{\tau} = -k_{154}\varphi_{HSO_3}^{n+1} + k_{149}\varphi_{SO_2}^{n} + f_{HSO_3}, \tag{18.14}$$

where $\alpha_{HSO_3} = -k_{154}$, $\beta = k_{149}\varphi_{SO_2}$. Concentrations of other harmful substances can be obtained in the same way.

18.4 Approximation, Stability and Convergence of the Difference Scheme

In order to consider approximation and stability, we integrate the left parts of the equations of motion, dividing by $h_1 h_2 h_3$ and obtain the following difference relations:

$$\begin{aligned}&\frac{1}{h_1h_2h_3}\int_{x_i}^{x_{i+1}}\int_{y_{j_{1/2}}}^{y_{j+1/2}}\int_{z_{k_{1/2}}}^{z_{k+1/2}}[\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x_1}+\frac{\partial uv}{\partial x_2}+\frac{\partial u\omega}{\partial x_3}+\frac{\partial p}{\partial x_1}]dx_1dx_2dx_3\\ &= (u^n_{t,i+1/2,j,k})+(u^2_{i+1,j,k}-u^2_{i,j,k})/h_1+[(uv)_{i+1/2,j+1/2.k}-(uv)_{i+1/2,j-1/2,k}]/h_2\\ &\quad+[(u\omega)_{i+1/2,j,k+1/2}-(u\omega)_{i+1/2,j,k-1/2}]/h_3+(p_{i+1,j,k}-p_{i,j,k})/h_1\end{aligned} \tag{18.15}$$

$$\begin{aligned}&\frac{1}{h_1h_2h_3}\int_{x_{i_{1/2}}}^{x_{i+1/2}}\int_{y_j}^{y_{j+1}}\int_{z_{k_{1/2}}}^{z_{k+1/2}}[\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x_1}+\frac{\partial v^2}{\partial x_2}+\frac{\partial v\omega}{\partial x_3}+\frac{\partial p}{\partial x_2}]dx_1dx_2dx_3\\ &= (v^n_{t,i,j+1/2,k})+[(uv)_{i+1/2,j+1/2.k}-(uv)_{i-1/2,j+1/2,k}]/h_1+(v^2_{i,j+1,k}-v^2_{i,j,k})/h_2\\ &\quad+[(v\omega)_{i,j+1/2,k+1/2}-(v\omega)_{i,j+1/2,k-1/2}]/h_3+(p_{i,j+1,k}-p_{i,j,k})/h_2\end{aligned} \tag{18.16}$$

$$\begin{aligned}&\frac{1}{h_1h_2h_3}\int_{x_{i_{1/2}}}^{x_{i+1/2}}\int_{y_{j_{1/2}}}^{y_{j+1/2}}\int_{z_k}^{z_{k+1}}[\frac{\partial \omega}{\partial t}+\frac{\partial u\omega}{\partial x_1}+\frac{\partial v\omega}{\partial x_2}+\frac{\partial \omega^2}{\partial x_3}+\frac{\partial p}{\partial x_3}]dx_1dx_2dx_3\\ &= (\omega^n_{t,i,j,k+1/2})+[(u\omega)_{i+1/2,j.k+1/2}-(u\omega)_{i+1/2,j,k+1/2}]/h_1\\ &\quad+[(v\omega)_{i,j+1/2,k+1/2}-(v\omega)_{i,j-1/2,k+1/2}]/h_2+(\omega^2_{i,j,k+1}-\omega^2_{i,j,k})/h_3\\ &\quad+(p_{i,j,k+1}-p_{i,j,k})/h_3\end{aligned} \tag{18.17}$$

Similarly, it is possible to obtain difference relations for the right parts, thus the following difference scheme is constructed:

$$\frac{u_{i+1/2,j,k}^{n+1} - u_{i+1/2,j,k}^{n}}{\tau} + L_{1,h}^{(1)} u_{i+1/2,j,k}^{n} + P_{x_1,i,j,k}^{n+1}$$
$$= \frac{1}{De} v_{i,j+1/2,k}^{n} + \frac{1}{Re_T}[(a_{i+1,j,k} u_{x_1,i+1/2,j,k}^{n})_{\overline{x}_1} + (a_{i+1/2,j+1/2,k} u_{x_2,i+1/2,j,k}^{n})_{\overline{x}_2}$$
$$+ (a_{i+1/2,j,k+1/2} u_{x_3,i+1/2,j,k}^{n})_{\overline{x}_3}]$$
$$+ f_{i+1/2,j,k}^{0}, \quad i = \overline{1, N_1 - 2},\ j = \overline{1, N_2 - 1}, k = \overline{1, N_3 - 1} \tag{18.18}$$

$$\frac{v_{i,j+1/2,k}^{n+1} - u_{i,j+1/2,k}^{n}}{\tau} + L_{1,h}^{(2)} v_{i,j+1/2,k}^{n} + P_{x_2,i,j,k}^{n+1}$$
$$= -\frac{1}{De} u_{i+1/2,j,k}^{n} + \frac{1}{Re_T}[(a_{i+1/2,j+1/2,k} v_{x_1,i+1/2,j+1/2,k}^{n})_{\overline{x}_1}$$
$$+ (a_{i,j+1,k} u_{x_2,i,j,k}^{n})_{\overline{x}_2} + (a_{i,j+1/2,k+1/2} v_{x_3,i,j+1/2,k+1/2}^{n})_{\overline{x}_3}]$$
$$+ f_{i,j+1/2,k}^{0}, \quad i = \overline{1, N_1 - 1},\ j = \overline{1, N_2 - 2}, k = \overline{1, N_3 - 1} \tag{18.19}$$

$$\frac{\omega_{i,j,k+1/2}^{n+1} - \omega_{i,j,k+1/2}^{n}}{\tau} + L_{1,h}^{(3)} \omega_{i,j,k+1/2}^{n} + P_{x_3,i,j,k}^{n+1} = -\overline{\lambda}\theta$$
$$+ \frac{1}{Re_T}[(a_{i+1/2,j,k+1/2} \omega_{x_1,i+1/2,j,k+1/2}^{n})_{\overline{x}_1} + (a_{i,j+1/2,k} \omega_{x_2,i,j+1/2,k}^{n})_{\overline{x}_2}$$
$$+(a_{i,j,k+1} \omega_{x_3,i,j,k}^{n})_{\overline{x}_3}] + f_{i,j,k+1/2}^{0}, i = \overline{1, N_1 - 1},\ j = \overline{1, N_2 - 1}, k = \overline{1, N_3 - 2}, \tag{18.20}$$

where

$$a_{i+1,j,k} = \int_{x_{2,j-1/2}}^{x_{2,j+1/2}} \int_{x_{3,k-1/2}}^{x_{3,k+1/2}}, a(x_{1,i+1}, x_2, x_3) dx_2 dx_3 \tag{18.21}$$

$$a_{i,j+1,k} = \int_{x_{1,i-1/2}}^{x_{1,i+1/2}} \int_{x_{3,k-1/2}}^{x_{3,k+1/2}} a(x_1, x_{2,j+1}, x_3) dx_1 dx_3, \tag{18.22}$$

and so on. Approximation of continuity equation is determined as:

$$div_h \overline{V}^{n+1} = u_{\overline{x}_1,i+1/2,j,k}^{n+1} + v_{\overline{x}_2,i,j+1/2,k}^{n+1} + \omega_{\overline{x}_3,i,j,k+1/2}^{n+1} = 0. \tag{18.23}$$

For an unambiguous determination of pressure, we require that the following equality be fulfilled:

$$\sum_{\overline{x}\in\Omega_h^{(1)}} p(\overline{x})h_1h_2h_3 = 0,\ \Omega_h^{(1)} \subseteq \Omega_h \tag{18.24}$$

Here, we use the following lemma in order to obtain the energy inequality:

Lemma 18.1 *For any grid functions $u_{i+1/2,j,k} \in \Omega_x$, $v_{i,j+1/2,k} \in \Omega_y$, $\omega_{i,j,k+1/2} \in \Omega_z$ satisfying conditions (18.23), (18.24), the identities are valid*

$$(L_{1,h}^{(1)}u_{i+1/2,j,k}, u_{i+1/2,j,k}) = (L_{1,h}^{(2)}v_{i,j+1/2,k}, v_{i,j+1/2,k})$$
$$= (L_{1,h}^{(3)}\omega_{i,j,k+1/2}, \omega_{i,j,k+1/2}) = 0, \tag{18.25}$$

where summation is performed by the internal nodes of the grid $\Omega_x \cup \Omega_y \cup \Omega_z$.

Defining the norm of the velocity vector as follows:

$$\| \overrightarrow{V}^n \|^2 = \sum_{\Omega_x}(u^n_{i+1/2,j,k})^2h_1h_2h_3 + \sum_{\Omega_y}(v^n_{i,j+1/2,k})^2h_1h_2h_3$$
$$+ \sum_{\Omega_z}(u^n_{i,j,k+1/2})^2h_1h_2h_3 \tag{18.26}$$

Multiplying difference equations (18.18)–(18.20) by $2\tau u^{n+1}_{i+1/2,j,k}h_1h_2h_3$, $2\tau v^{n+1}_{i,j+1/2,k}h_1h_2h_3$ and $2\tau\omega^{n+1}_{i,j,k+1/2}h_1h_2h_3$ respectively, and then summing up by points Ω_x, Ω_y, Ω_z, and by making some transformations, we obtain the following basic energy inequality:

$$\| \overrightarrow{V}^{n+1} \|^2 - \| \overrightarrow{V}^n \|^2 + \| \overrightarrow{V}^{n+1} - \overrightarrow{V}^n \|^2 + 2\tau(L_{1h}\overrightarrow{V}^n, V^{n+1})$$
$$+2\tau(\sum_{\Omega_x} p^{n+1}_{x_1}u^{n+1}_{i+1/2,j,k} + \sum_{\Omega_y} p^{n+1}_{x_2}v^{n+1}_{i,j+1/2,k} + \sum_{\Omega_y} p^{n+1}_{x_3}\omega^{n+1}_{i,j,k+1/2})h_1h_2h_3 + 2\tau d_n$$
$$\leq \frac{2\tau}{DE}|S_h| + 2\tau|(\overrightarrow{f}^n, \overrightarrow{V}^{n+1})|, \tag{18.27}$$

where

$$d_h = -(\sum_{\Omega_x} L^{(1)}_{2h}u^n_{i+1/2,j,k}u^{n+1}_{i+1/2,j,k}$$
$$+ \sum_{\Omega_y} L^{(2)}_{2h}v^n_{i,j+1/2,k}v^{n+1}_{i,j+1/2,k} + \sum_{\Omega_y} p^{n+1}_{x_3}L^3_{2h}\omega^n_{i,j,k+1/2}\omega^{n+1}_{i,j,k+1/2})h_1h_2h_3.$$

$$(\vec{f}^{\,n}, \vec{V}^{\,n+1}) = \sum_{\Omega_h}(f^0_{i+1/2,j,k}u^{n+1}_{i+1/2,j,k} + f^0_{i,j+1/2,k}v^{n+1}_{i,j+1/2,k} + f^0_{i,j,k+1/2}\omega^{n+1}_{i,j,k+1/2})h_1h_2h_3 \tag{18.28}$$

$$\begin{aligned} S_h &= \sum_{\Omega_x} v^n_{i,j+1/2,k}u^{n+1}_{i+1/2,j,k}h_1h_2h_3 - \sum_{\Omega_y} u^n_{i+1/2,j,k}v^{n+1}_{i,j+1/2,k}h_1h_2h_3 \\ &= \sum_{\Omega_x} v^n_{i,j+1/2,k}(u^{n+1}_{i+1/2,j,k} - u^n_{i+1/2,j,k})h_1h_2h_3 \\ &\quad - \sum_{\Omega_y} u^n_{i+1/2,j,k}(v^{n+1}_{i,j+1/2,k} - v^n_{i,j+1/2,k})h_1h_2h_3. \end{aligned} \tag{18.29}$$

After some transformations, we obtain estimate of the following form:

$$\begin{aligned} \| \vec{V}^{\,n+1} \|^2 + C_1 \sum_{k=0}^{n} \| \nabla_h \vec{V}^{\,n+1} \|^2 \le \| \vec{V}^{\,0} \|^2 + 2\tau (\sum_{k=0}^{n} \| \vec{f}^{\,k} \|)(\| \vec{V}^{\,0} \| \\ + 2\tau \sum_{k=0}^{n} \| \vec{f}^{\,k} \|) \le 2 \| \vec{V}^{\,0} \|^2 + 5(\tau \sum_{k=0}^{n} \| \vec{f}^{\,k} \|)^2. \end{aligned} \tag{18.30}$$

To examine the convergence of the finite-difference solution to that of the differential problem, we analyze the finite-difference relations corresponding to the equations of the model (18.1)–(18.6).

To consider the convergence of the solution of a finite-difference problem, the following theorem will take place:

Theorem 18.1 *Let the conditions* $C_1 - C_3 \left\|\boldsymbol{\phi}^n\right\| \left\|\nabla_h \boldsymbol{\phi}^n\right\| \ge 0$; $1 - \frac{24\tau C_1}{h^2} - \frac{2}{\tau} - \frac{2\tau}{De} > 0$; $1 - 2\tau C_4 > 0$ *be fulfilled, then the solution of difference scheme of model (1)–(7) is stable and converges to the solution of differential problem (1)–(7) at a rate determined by the inequality:*

$$\left\|\boldsymbol{\phi}^{n+1}\right\|^2 + C_5 \sum_{k=0}^{n} \left\|\nabla_h \boldsymbol{\phi}^{n+1}\right\|^2 \le C_6(\tau^2 + h^4).$$

18.5 Numerical Results

Numerical calculations were performed for an area of 35×35 km and a fixed surface layer height of 3500 m. The numerical implementation was performed based on the following data: convection parameter (λ) has been taken as 0.16 $\mathrm{m(s \cdot deg)^{-1}}$, Coriolis force ($l$) as $10^{-4}\,\mathrm{s^{-1}}$, both horizontal (μ_{x_1}) and vertical (μ_{x_2}) coeff. of turbulent exchange as $6 \cdot 10^3\,\mathrm{m^2\,s^{-1}}$, characteristic length scale (L) as 35 km, wind speed (U) as $10\,\mathrm{m \cdot s^{-1}}$, speed temperature ($\theta$) as 20 °C (Fig. 18.1).

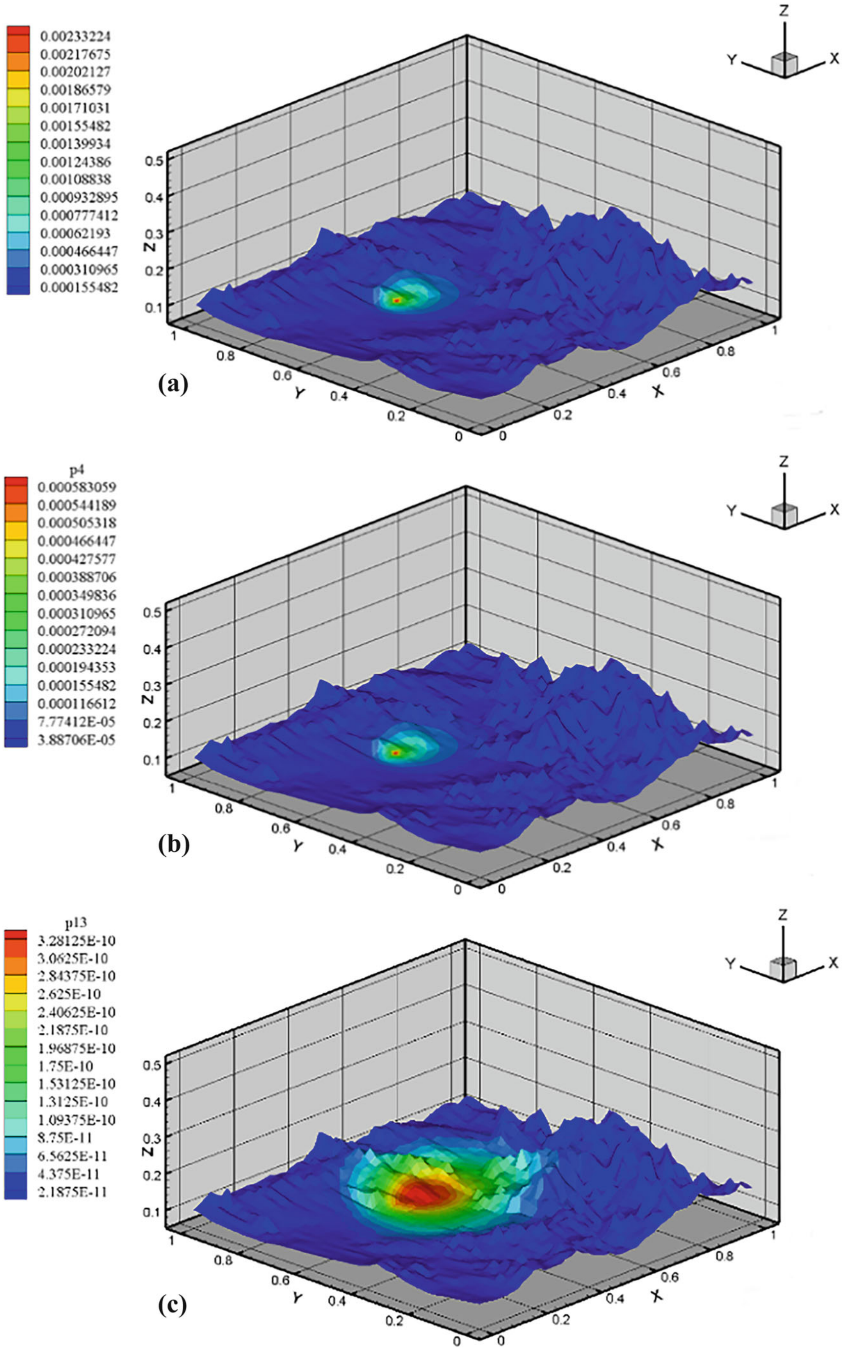

Fig. 18.1 Spread of amount of harmful substances CO (**a**), SO_2 (**b**) and H_2SO_4 (**c**), respectively

18.6 Conclusion

This study contributes to our understanding of pollutant dispersion in the atmospheric air of Ust-Kamenogorsk. Employing the atmospheric boundary layer model, we successfully modeled and predicted the spread of harmful substances, considering complex multidimensional non-stationary problems. The considered finite difference method and splitting scheme for physical processes provided effective difference schemes for the model, demonstrating satisfactory approximation, stability, and convergence. Based on the considered model, software is being developed to simulate and predict the distribution of harmful substances.

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References

1. Temirbekov, N., Kasenov, S., Berkinbayev, G., Temirbekov, A., Tamabay, D., Temirbekova, M.: Analysis of data on air pollutants in the city by machine–intelligent methods considering climatic and geographical features. Atmosphere **14**(5), 892 (2023)
2. Marchuk, G.I.: Mathematical Modeling in the Problem of the Environment. Moscow, Nauka (1982) (in Russian)
3. Penenko, V.V., Aloyan, A.E.: Models and Methods for Environmental Protection Problems. Novosibirsk, Nauka (1985)
4. Aloyan, A.E., Yermakov, A.N., Arutyunyan, V.O.: Modeling the convective cloudiness and its impact on the atmospheric gaseous composition. News Atmos. Ocean. Phys. **6**(46), 713–726 (2010)
5. Arguchintsev, V.K.: Dynamics of the Atmosphere, vol. 130. Irkutsk State University, Irkutsk (2006) (in Russian)
6. Olsen, H.R., Loftstrom, P., Berkowicz, R., Jensen, A.B.: An improved dispersion model for regulatory use – the OML model. In: Air Pollution Modeling and its Application IX, pp. 29–38 (1992)
7. Zakarin, E., Baklanov, A., Balakay, L., Dedova, T., Bostanbekov, K.: Modeling of the calm situations in the atmosphere of Almaty. Asian J. Atmos. Environ. **16**(1), 1–15 (2022)
8. Zakarin, E., Baklanov, A., Balakay, L., Dedova, T., Bostanbekov, K.: Simulation of air pollution in Almaty City under adverse weather conditions. Russian Meteorol. Hydrol. **46**(2), 121–128 (2021)
9. Abdibekov, U.S., Zhumagulov, B.T., Hikmetov, A.K.: Modeling of impurity propagation in a free atmosphere. Comput. Technol. **8**, 25–35 (2003)
10. Issakhov, A., Omarova, P., Issakhov, A.: Numerical study of thermal influence to pollutant dispersion in the idealized urban. Air Qual. Atmos. Health **13**, 1045–1056 (2020)
11. Danaev, N.T., Isakhov, A.A., Abdibekov, A.U., Hikmetov, A.K.: Modeling of impurity transfer in a stratified medium by the method of large vortices. Mining Inf. Anal. Bull. **17**, 73–78 (2009) (in Russian)
12. Malgazhdarov, Y., Wojcik, W., Adikanova, S., Madiyarov, M.N., Myrzagalieva, A.B., Temirbekov, N.M., Junisbekov, M., Pavlovsky, L.: Probabilistic and statistical modeling of harmful transport impurities in the atmosphere from motor transport. J. "Chronol. Affin." **19**, 795–808 (2017)

13. Smagulov, Sh.S., Temirbekov, N.M., Kamaubaev, K.S.: Modeling by the fictitious domain method of the boundary condition for pressure in fluid flow problems. Siberian J. Comput. Math. **3**(1), 57–71 (2000)
14. Zhumagulov, B.T., Smagulov, S.S., Danaev, N.T., Kuznetsov B.G.: Investigation of convergence of economic finite difference schemes of the Navier-Stokes equation in variables (u,v,p). Model. Mech. Novosib. **6**(23), 2, 25–57 (1992) (in Russian)
15. Temirbekov, A.N., Danaev, N.T., Malgazhdarov, E.A.: Modeling of pollutants in the atmosphere based on photochemical reactions. Eurasian Chemico-Technol. J. **16**(1), 61–71 (2014)
16. Temirbekov, A.N., Urmashev, B.A., Gromaszek, K.: Investigation of the stability and convergence of difference schemes for the three-dimensional equations of the atmospheric boundary layer. Int. J. Electron. Telecommun. **3**(64), 391–396 (2018)
17. Temirbekov, N., Malgazhdarov, Y., Tamabay, D., Temirbekov, A.: Atmospheric modelling of photochemical transformations of pollutants: Impact of weather conditions and diurnal cycle (Case study: Ust-Kamenogorsk, Kazakhstan). Math. Model. Eng. Probl. **10**(5), 1699–1705 (2023)
18. Temirbekov, N., Malgazhdarov, Y., Tamabay, D., Temirbekov, A.: Mathematical and computer modeling of atmospheric air pollutants transformation with input data refinement. Indonesian J. Electr. Eng. Comput. Sci. **32**(3), 1405–1416 (2023)

Chapter 19
Application of Adjoint Equations for Numerical Solution of Problems Using the Fictitious Domain Method

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Abstract The article describes a method for increasing the accuracy of numerical results of mathematical physics problems solved using the fictitious domain method (FDM). The one-dimensional Burgers problem with a curved boundary is considered as a model problem. To demonstrate the effectiveness of the proposed FDM variant, the original domain is extended to a simple auxiliary domain and the coefficients and the right-hand side of the equation are extended accordingly. At the boundary of the auxiliary domain boundary conditions are set to facilitate the numerical solution of the auxiliary FDM problem. In the classic version of FDM, there are differences between numerical and exact solutions of problems. In order to reduce this difference at the boundary of the original region, this article uses the variational method. The application of the variational method leads to the solution of the conjugate problem of the auxiliary FDM problem. To minimize the functional difference of the boundary condition on the original boundary an iterative process of the conjugate gradient method is constructed. Numerous calculations have been carried out over a wide range of input parameters, which show the efficiency and high accuracy of the proposed numerical method.

19.1 Introduction

When solving problems of mathematical physics using the finite difference method, the separate problem is the curvilinearity of the initial boundaries. One of the common methods for solving this problem is to use FDM. FDM is a method of simplifying a geometrically complex original region by replacing this region with a geometrically simple region, for example, a rectangle or parallelepiped [1–3].

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For example, work [4] considers FDM for numerical simulation of the flow of a viscous incompressible fluid in complex geometric domains. The problem is considered in a discretely given doubly connected domain with a curvilinear boundary.

In the studied variants of FDM the same physical boundary conditions were set on the boundary of the auxiliary domain as on the original solid boundary, i.e. the problem of a complex curved boundary was removed, but the problem of setting boundary conditions for pressure or total head remained.

In [5], an original version of FDM was proposed for the Navier-Stokes equations of a viscous incompressible fluid in variable fields of velocity and pressure. Moreover, at the boundary of the auxiliary region, boundary conditions are set for the pressure in the form of a constant and the tangential components of the velocity equal to zero. The solvability of the auxiliary problem of FDM with such boundary conditions is proved. The presence of a boundary condition is needed to obtain the Dirichlet problem for the Poisson equation for pressure. In this way, three problems are solved: the issue of a curvilinear boundary and modeling the boundary condition for pressure or total pressure, the issue of constructing homogeneous difference schemes.

At the same time, in this method the assumption is often made that the boundary condition on the fictitious boundary is equal to the boundary condition on the original boundary or, in some cases, is even zero. Thus, due to this assumption, FDM has the problem that the boundary condition at the original boundary is not satisfied.

The article proposes some addition to the above method to take into account in the numerical solution the boundary condition on the original boundary. Satisfying the boundary condition is considered as a variational problem. Minimizing the corresponding functional leads to the adjoint equation. Thus, the condition on the original boundary is found by the iterative method. This method is also described in article [6]. Also in articles [7–10] inverse problems are considered.

To illustrate the capabilities of the developed method the Burgers model initial-boundary value problem was considered. The problem of FDM and the conjugate problem are constructed. An algorithm for minimizing the functional is presented. Numerous methodological calculations have been carried out.

19.2 Solving the Problem Using the Fictitious Domain Method

The initial-boundary value problem for the one-dimensional Burgers equation is considered in the domain $Q_T = (0, \xi) \times (0, T),\ 0 < \xi < 1$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \mu\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in Q_T, \tag{19.1}$$

$$u(0,t) = g_1(t), \quad t \in (0,T), \tag{19.2}$$

$$u(\xi,t) = g_2(t), \quad t \in (0,T), \tag{19.3}$$

$$u(x,0) = v(x), \quad x \in (0,\xi). \tag{19.4}$$

To demonstrate the methodology of the proposed FDM variant, task (19.1)–(19.4) was continued into a fictitious subdomain and the auxiliary problem was considered in the domain $Q_T^\varepsilon = (0,1) \times (0,T)$:

$$\frac{\partial u^\varepsilon}{\partial t} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x} - \frac{\partial}{\partial x}(\mu^\varepsilon \frac{\partial u^\varepsilon}{\partial x}) = f^\varepsilon(x,t), \quad (x,t) \in Q_T^\varepsilon, \tag{19.5}$$

$$u^\varepsilon(0,t) = g_1(t), \quad t \in (0,T), \tag{19.6}$$

$$u^\varepsilon(1,t) = 0, \quad t \in (0,T), \tag{19.7}$$

$$u^\varepsilon(x,0) = v^\varepsilon(x), \quad x \in (0,1), \tag{19.8}$$

$$\mu^\varepsilon(x) = \begin{cases} \mu, & 0 < x < \xi \\ \mu/\varepsilon, & \xi < x < 1, \end{cases}$$

$$f^\varepsilon(x,t) = \begin{cases} f(x,t), & (x,t) \in Q_T \\ 0, & (x,t) \in Q_T^\varepsilon / Q_T, \end{cases}$$

$$v^\varepsilon(x) = \begin{cases} v(x), & 0 < x < \xi \\ 0, & \xi < x < 1. \end{cases} \tag{19.9}$$

In the above statement of the problem, the boundary condition at the actual boundary is not taken into account. Thus, the following inherent in the FDM assumption is made: outside the original domain but within the fictitious domain, the viscosity coefficient increases significantly, and there is no flow. Thus, at the boundary of the fictitious domain, the flow velocity is zero.

Below are the calculation results for the exact solution in the form $u = t + sin\pi x$:

As it can be seen from Figs. 19.1 and 19.2, the results of the numerical solution by FDM and the exact solution are very close to each other. At the same time, when comparing in digital indicators, there is some difference between the indicated solutions, which is noticeable in the lower right corner of Fig. 19.3. It is supposed that the reason for this difference is the assumption of the boundary condition. Next, it is proposed to increase the accuracy of the numerical solution to the problem by taking into account the boundary condition on the actual boundary.

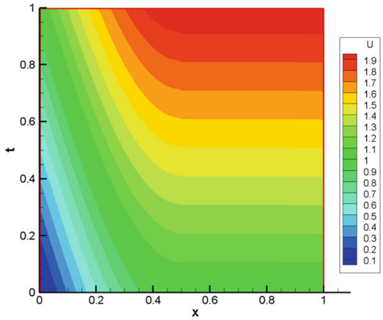

Fig. 19.1 Numerical result of $u(x,t)$

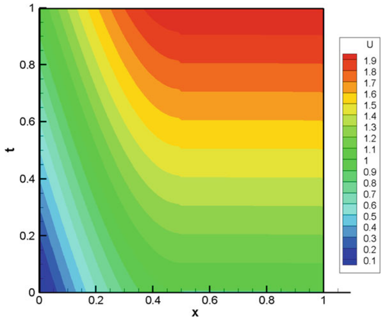

Fig. 19.2 Graph of $u = t + sin\pi x$

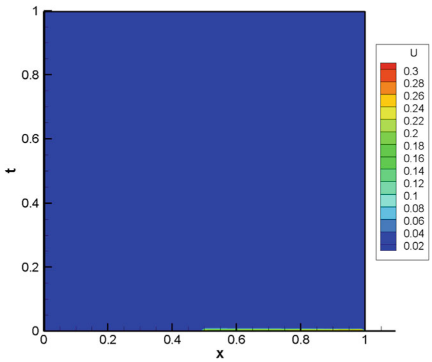

Fig. 19.3 Difference between numerical and exact result of $u(x,t)$

19.3 Using the Adjoint Problem to Improve the Accuracy of the Numerical Solution When Using FDM

To increase the accuracy of the numerical solution when using FDM it is proposed to take as a solution to the problem that series of velocities whose value at the point $x = \xi$ is very close to $g_2(t)$, for which it is proposed to minimize the following functional:

$$J(g_2) = \int_0^T (u^{\varepsilon}(\xi,t) - g_2(t))^2 dt. \tag{19.10}$$

The Lagrangian $L(g_2)$ is defined for the minimization problem:

$$L(g_2) = \int_0^T (u^\varepsilon(\xi, t) - g_2(t))^2 dt$$
$$+ \int_0^T \int_0^1 \psi(\frac{\partial u^\varepsilon}{\partial t} + u^\varepsilon \frac{\partial u^\varepsilon}{\partial x} - \frac{\partial}{\partial x}(\mu^\varepsilon \frac{\partial u^\varepsilon}{\partial x}) - f^\varepsilon(x, t))dxdt, \tag{19.11}$$

where $\psi(x, t)$ is the Lagrange multiplier. The second term is the condition according to which it is necessary to minimize the functional $J(g_2)$. This condition is taken from Eq. (19.5).

To minimize the functional, the following increment is constructed:

$$L(g_2 + \delta g_2) - L(g_2) = \delta L =< \delta g_2, L'(g_2) > . \tag{19.12}$$

The following notation is used for this:

$$u^\varepsilon(x, t; g_2 + \delta g_2) = \tilde{u}^\varepsilon, \tag{19.13}$$

$$u^\varepsilon(x, t; g_2) = u^\varepsilon, \tag{19.14}$$

$$\tilde{u}^\varepsilon - u^\varepsilon = \delta u^\varepsilon.$$

The perturbed problem is formulated:

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t} + \tilde{u}^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x} - \frac{\partial}{\partial x}(\mu^\varepsilon \frac{\partial \tilde{u}^\varepsilon}{\partial x}) = f^\varepsilon(x, t), \quad (x, t) \in Q_T,$$

$$\tilde{u}^\varepsilon(0, t) = g_1, \quad t \in (0, T),$$

$$\tilde{u}^\varepsilon(\xi, t) = g_2 + \delta g_2, \quad t \in (0, T),$$

$$\tilde{u}^\varepsilon(1, t) = 0, \quad t \in (0, T),$$

$$\tilde{u}^\varepsilon(x, 0) = v^\varepsilon(x), \quad x \in (0, 1).$$

From the problem (19.7)–(19.9), the problem (19.5)–(19.8) is subtracted to obtain the problem for δu^ε:

$$\frac{\partial \delta u^\varepsilon}{\partial t} + \delta u^\varepsilon \frac{\partial u^\varepsilon}{\partial x} + u^\varepsilon \frac{\partial \delta u^\varepsilon}{\partial x} - \frac{\partial}{\partial x}(\mu^\varepsilon \frac{\partial \delta u^\varepsilon}{\partial x}) = 0, \tag{19.15}$$

$$\delta u^\varepsilon(0, t) = 0, \quad t \in (0, T), \tag{19.16}$$

$$\delta u^\varepsilon(\xi, t) = \delta g_2, \quad t \in (0, T), \tag{19.17}$$

$$\delta u^{\varepsilon}(1,t) = 0, \quad t \in (0,T),$$

$$\delta u^{\varepsilon}(x,0) = 0, \quad x \in (0,1).$$

Thus, the Lagrangian increment is as follows:

$$L(g_2 + \delta g_2) - L(g_2) = \delta L = \int_0^T (u^{\varepsilon}(\xi,t) - g_2(t))\delta u^{\varepsilon}(\xi,t)dt$$
$$+ \int_0^T \int_0^1 \psi(\frac{\partial \delta u^{\varepsilon}}{\partial t} + \frac{\partial (u^{\varepsilon}\delta u^{\varepsilon})}{\partial x} - \frac{\partial}{\partial x}(\mu^{\varepsilon}\frac{\partial \delta u^{\varepsilon}}{\partial x}))dxdt.$$

The identically zero expression from (19.10) is considered:

$$0 = \int_0^T \int_0^1 \psi(\frac{\partial \delta u^{\varepsilon}}{\partial t} + \frac{\partial (u^{\varepsilon}\delta u^{\varepsilon})}{\partial x} - \frac{\partial}{\partial x}(\mu^{\varepsilon}\frac{\partial \delta u^{\varepsilon}}{\partial x}))dxdt.$$

This expression is integrated in three parts: Integrating the first term, we get

$$\int_0^T \int_0^1 \psi(\frac{\partial \delta u^{\varepsilon}}{\partial t})dxdt = \int_0^1 (\psi \delta u^{\varepsilon}|_0^T - \int_0^T \delta u^{\varepsilon}(\frac{\partial \psi}{\partial t})dt)dx.$$

Integrating the second term, we get

$$\int_0^T \int_0^1 \psi(\frac{\partial (u^{\varepsilon}\delta u^{\varepsilon})}{\partial x})dxdt = \int_0^T (\psi u^{\varepsilon}\delta u^{\varepsilon}|_0^1 - \int_0^1 u^{\varepsilon}\delta u^{\varepsilon}(\frac{\partial \psi}{\partial x})dx)dt.$$

Integrating the third term, we get

$$\int_0^T \int_0^1 \psi \frac{\partial}{\partial x}(\mu^{\varepsilon}\frac{\partial \delta u^{\varepsilon}}{\partial x})dxdt = \int_0^T (\psi \mu^{\varepsilon}\frac{\partial \delta u^{\varepsilon}}{\partial x}|_0^{\xi} + \psi \mu^{\varepsilon}\frac{\partial \delta u^{\varepsilon}}{\partial x}|_{\xi}^1$$
$$-\frac{\partial \psi}{\partial x}\mu^{\varepsilon}\delta u^{\varepsilon}|_0^{\xi} - \frac{\partial \psi}{\partial x}\mu^{\varepsilon}\delta u^{\varepsilon}|_{\xi}^1 + \int_0^1 \delta u^{\varepsilon}\frac{\partial}{\partial x}(\mu^{\varepsilon}\frac{\partial \psi}{\partial x})dx)dt.$$

Thus, the Lagrangian increment has the following form:

$$\delta L = \int_0^T (u^{\varepsilon}(\xi,t) - g_2(t))\delta u^{\varepsilon}(\xi,t)dt$$
$$-\int_0^T \int_0^1 \delta u^{\varepsilon}(\frac{\partial \psi}{\partial t} + u^{\varepsilon}\frac{\partial \psi}{\partial x} + \frac{\partial}{\partial x}(\mu^{\varepsilon}\frac{\partial \psi}{\partial x}))dxdt$$
$$+\int_0^1 (\psi(x,T)\delta u^{\varepsilon}(x,T) - \psi(x,0)\delta u^{\varepsilon}(x,0))dx$$

$$+\int_0^T (\psi(1,t)u^\varepsilon(1,t)\delta u^\varepsilon(1,t) - \psi(0,t)u^\varepsilon(0,t)\delta u^\varepsilon(0,t))dt$$

$$-\int_0^T (\psi(\xi-0,t)\mu^\varepsilon(\xi-0)\frac{\partial \delta u^\varepsilon(\xi-0,t)}{\partial x} - \psi(0,t)\mu^\varepsilon(0)\frac{\partial \delta u^\varepsilon(0,t)}{\partial x})dt$$

$$-\int_0^T (\psi(1,t)\mu^\varepsilon(1)\frac{\partial \delta u^\varepsilon(1,t)}{\partial x} - \psi(\xi+0,t)\mu^\varepsilon(\xi+0)\frac{\partial \delta u^\varepsilon(\xi+0,t)}{\partial x})dt$$

$$+\int_0^T (\frac{\partial \psi(\xi-0,t)}{\partial x}\mu^\varepsilon(\xi-0)\delta u^\varepsilon(\xi-0,t) - \frac{\partial \psi(0,t)}{\partial x}\mu^\varepsilon(0)\delta u^\varepsilon(0,t))dt$$

$$+\int_0^T (\frac{\partial \psi(1,t)}{\partial x}\mu^\varepsilon(1)\delta u^\varepsilon(1,t) - \frac{\partial \psi(\xi+0,t)}{\partial x}\mu^\varepsilon(\xi+0)\delta u^\varepsilon(\xi+0,t))dt.$$

Using (19.11)–(19.14), the formulation of the conjugate problem is as follows:

$$\frac{\partial \psi}{\partial t} + u^\varepsilon \frac{\partial \psi}{\partial x} + \frac{\partial}{\partial x}(\mu^\varepsilon \frac{\partial \psi}{\partial x}) = (u^\varepsilon(x,t) - g_2(t))\delta(x-\xi)$$

$$\psi(0,t) = 0, \psi(1,t) = 0, \psi(x,T) = 0,$$

$$\psi(\xi-0,t)\mu^\varepsilon(\xi-0)\frac{\partial g_2(\xi-0,t)}{\partial x} = \psi(\xi+0,t)\mu^\varepsilon(\xi+0)\frac{\partial g_2(\xi+0,t)}{\partial x},$$

$$(\psi\mu^\varepsilon(g_2)_x)_{x=\xi} = 0.$$

Thus, the Lagrangian increment has the following form:

$$\delta L =< \delta g_2, L'(g_2) >= \int_0^T (\frac{\partial \psi(\xi-0,t)}{\partial x}\mu^\varepsilon(\xi-0) - \frac{\partial \psi(\xi+0,t)}{\partial x}\mu^\varepsilon(\xi+0))\delta g_2 dt.$$

The Lagrangian gradient at g_2 is as follows:

$$L'(g_2) = \frac{\partial \psi(\xi-0,t)}{\partial x}\mu^\varepsilon(\xi-0) - \frac{\partial \psi(\xi+0,t)}{\partial x}\mu^\varepsilon(\xi+0).$$

Next, using the gradient and the initial approximation, an appropriate value of $u^\varepsilon(\xi,t)$, at which the Lagrangian approaches a minimum, is selected by iterative methods. The entire series of velocity values, which contains a more suitable value $u^\varepsilon(\xi,t)$, is assumed to be a more accurate solution to the problem (19.5)–(19.9).

Below are the difference schemes of the direct and conjugate problems:

1. To numerically solve the problem (19.5)–(19.9), the "upwind" difference scheme is used in the implicit form:

$$u_{t,i}^{n}+\frac{1}{2}[(u_i^n-|u_i^n|)\cdot u_{x,i}^{n+1}+(u_i^n+|u_i^n|)\cdot u_{\bar{x},i}^{n+1}]=(\mu_{i+1/2}\cdot u_{\acute{x},i}^{n+1})_x+f_i^n,$$

$$i=\overline{1,N_x-1},\ n=\overline{1,N_t-1},$$

$$u_0^n=g_1^n,\ u_{NN}^n=g_2^n,\ u_N^n=0,\ n=\overline{0,N_t},\ x_{NN}=\xi,u_i^0=v_i,\ i=\overline{0,N_x}.$$

2. Discretization of the adjoint problem (19.15)–(19.17):

$$\psi_{t,i}^{n}+u_i^n\cdot\psi_{\mathring{x},i}^{n-1}+(\mu_{i+1/2}\cdot\psi_{\acute{x},i}^{n-1})_x=u_{NN}^n-g_2^n$$

$$i=\overline{1,N_x-1},\ n=\overline{N_t-1,1},\ x_{NN}=\xi$$

$$\psi_0^n=0,\ \psi_N^n=0,\ n=\overline{0,N_t},\psi_i^{N_t}=0,\ i=\overline{0,N_x}.$$

The results of minimizing the functional when an exact solution is in the form $u=t+sin\pi x$ are as in Figs. 19.4, 19.5, and 19.6.

Based on the above graphs, there is a significant decrease in the target functional and the difference between the numerical and exact value of the velocity at the actual boundary.

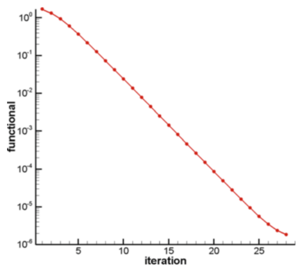

Fig. 19.4 Reduction of functional

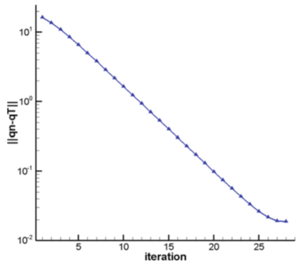

Fig. 19.5 Numerical and exact boundary condition difference at $x=\xi$

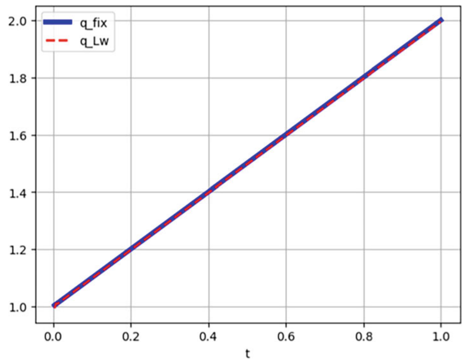

Fig. 19.6 Numerical and exact velocity values at the actual boundary

19.4 Conclusion

Thus, it is noted that the target functional is minimized and the boundary condition on the actual boundary is taken into account. At the same time, during the numerical implementation of the above algorithm, it was noted that the time spent on solving the problem taking into account the boundary condition significantly exceeds the time spent on solving the problem without minimizing the functional.

References

1. Smagulov, Sh.S., Orunkhanov, M.K.: Fictitious domain method for the Navier-Stokes equations with inhomogeneous boundary conditions. Math. Model. **12**(10), 121–127 (2000) (in Russian)
2. Temirbekov, A.N., Zhaksylykova, Zh.R., Malgazhdarov, Y.A., Kasenov, S.E.: Application of the fictitious domain method for Navier-Stokes equations. Comput. Mater. Contin. **73**(1), 2035–2055 (2022)
3. Temirbekov, A.N., Kasenov, S.E., Temirbekova, L.N.: Fictitious domain method for atmosphere boundary layer model. AIP Conf. Proc. **2483**, 060009 (2022)
4. Temirbekov, N.M., Tokanova, S.O., Malgazhdarov, Y.A.: Information technology for numerical simulation of viscous incompressible flow in biconnected domains. J. Theor. Appl. Inf. Technol. **88**, 441–448 (2016)
5. Smagulov, Sh.S., Temirbekov, N.M., Danayev, N.T.: Modeling boundary conditions for pressure and total head in hydrodynamics problems using the fictitious domain method (in Russian). Rep. Acad. Sci. **374**(3), 333–335 (2000)
6. Temirbekov, A.N., Temirbekova, L.N., Zhumagulov, B.T.: Fictitious domain method with the idea of conjugate optimization for non-linear Navier-Stokes equations. Appl. Comput. Math. **22**(2), 172–188 (2023)
7. Azimov, A.A, Kasenov, S.E., Nurseitov, D.B., Serovajsky, S.Y.: Inverse problem for the Verhulst equation of limited population growth with discrete experiment data. AIP Conf. Proc. **1759**, 020037 (2016)

8. Kasenov, S.E., Urmashev, B.A., Temirbekov, A.N., Amantayeva, A.B.: Numerical solution of the inverse pharmacokinetic problem for the three-compartment model. J. Eng. Sci. Technol. Rev., (Special Issue), 122–126 (2020)
9. Kabanikhin, S.I., Nurseitova, A., Kasenov, S.E.: Stability estimation of the generalized solution to the direct problem for the acoustic equation. J. Phys. Conf. Ser. **2092**(1), 012005 (2021)
10. Kasenov, S.E., Askerbekova, J.A., Tleulesova, A.M.: Algorithm construction and numerical solution based on the gradient method of one inverse problem for the acoustics equation. Eastern-Eur. J. Enterprise Technol. **2**(5), 116, 43–52 (2022)

Chapter 20
Numerical Modeling of Diffusion Processes in Two-Component Nonlinear Media with Variable Density and Source

Mersaid Aripov and Dilobar Nigmanova

Abstract In this work self-similar and approximately self-similar approach to the study of solutions of a double nonlinear parabolic system with a variable density and a source or absorption. The estimate of the weak solution, free boundary and the asymptotic behavior of the self-similar solution of the system for different cases studied. The qualitative properties of a solution of reaction-diffusion system and a double non-linearity with variable density established. Based it the problem of initial approximation solved, numerical analysis of the Cauchy problem for slowly diffusion, fast diffusion, critical and singular cases are analyzed.

20.1 Introduction

Mathematical models generated by modern problems of science and technology are, as a rule, non-linear. One of the effective, universal methods for solving nonlinear problems is the technology of computational experiment. However, carrying out a computational experiment requires solving the problem of choosing the initial approximation, which has the effect of nonlinearity, which can be achieved, for example, by invariant group analysis of solutions.

In the domain $Q = \left\{(t, x) : \ \ 0 < t, \quad x \in R^N\right\}$ consider a reaction-diffusion system with a double nonlinearity and a variable density in a two-componential medium described by the following system of degenerate parabolic equations with variable density and time dependent nonlinear source or absorption

$$\begin{aligned} |x|^{-l}\frac{\partial u}{\partial t} &= \nabla\left(|x|^n u^{m_1-1}\left|\nabla u^k\right|^{p-2}\nabla u^{l_1}\right) + \varepsilon|x|^{-l}\gamma(t)u^{p_1}v^{q_1} = 0, \\ |x|^{-l}\frac{\partial v}{\partial t} &= \nabla\left(|x|^n v^{m_2-1}\left|\nabla v^k\right|^{p-2}\nabla v^{l_2}\right) + \varepsilon|x|^{-l}\gamma(t)u^{p_2}v^{q_2} = 0, \end{aligned} \tag{20.1.1}$$

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with initial condition

$$u(0,x) = u_0(x) \geq 0, \quad v(0,x) = v_0(x) \geq 0, \quad x \in R^N, \tag{20.1.2}$$

where, $k \in R$, $m_1, m_2 > 1$, $p_i, q_i \geq 1$, $p \geq 2$ are positive real numbers and $u_0(x) \geq 0, v_0(x) \geq 0$- are a non-trivial, non-negative, bounded and sufficiently smooth functions, $0 < \gamma(t) \in C(0,\infty)$.

System (20.1.1) describes various physical processes in a two-component non-linear medium of the reaction-diffusion, heat conduction, combustion, polytropic filtration of liquid and gas processes with variable density at the presence of a source or an absorption power of which is equal to $\gamma(t)u^{p_i}v^{q_i}$.

In the domain where $u = 0$, $v = 0$ or $\nabla u = 0$, $\nabla v = 0$, the system of equations (20.1.1) degenerates into first-order equations. Therefore, it is necessary to investigate a weak solution, since the system of equations (20.1.1) may not have a solution in the classical sense.Therefore in this case, the weak solution is considered in a class having a physical meaning. Notice that before numerically solving considered problem (20.1.1), it is necessary to study various qualitative properties, such as the finite speed of perturbation, localization of the solution, behavior of the front (a free boundary), asymptotes of the self-similar solutions depending on the values of the numerical parameters of the system of equations (20.1.1).

Problem (20.1.1), (20.1.2) and one equation case of it for the particular value of the numerical parameters have been studied intensively by numerous authors (see [1–7] and references therein). In particular, Samarsky A.A., Kurdyumov S.P. et al. [1] study a condition for the global solvability of the Cauchy problem for a degenerate parabolic system for the case where $p = 2, n = l = 0$. They developed the theory and practice of studying the blow up properties of solutions in the case. $p = 2$ or $m = 1, n = l = 0$. Special methods for studying blow up solutions of non-linear parabolic equations have been developed, which make it possible to carry out a sufficiently detailed study of blow up solutions using the example of a heat equation with a source of a general form. The property of global solvability and unsolvability for the case of one equation with variable density was studied by lot of authors in [2], where a condition of global solvability of the Fujita type was obtained [4].

In the works [3–7] studied the Cauchy problems to the following two equations with variable coefficients:

$$\begin{aligned}
&\rho_1(x)\, u_t = div\left(\rho_2 u^{m-1}|Du|^{\lambda-1}Du\right) + \rho_3(x)\, u^p \\
&(x,t) \in Q_T = \mathbb{R}^N \times (0,T), \quad T > 0, \quad N \geq 1 \\
&u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N
\end{aligned}$$

where $\lambda > 0$, $m+\lambda-2 > 0$, $p > m+\lambda-1$, $\rho_1(x) = |x|^l$, $\rho_2(x) = |x|^n$, $\rho_3(x) = |x|^q$. It was shown that under some restrictions on the parameters, any nontrivial solution to the Cauchy problem blows up in a finite time. Moreover, the authors established a sharp universal estimate of the solution near the blow-up point. They found conditions on the parameters of the problem under which the solution of the Cauchy problem explodes in a finite time. Moreover, an exact universal one has been obtained, i.e. independent of the initial function, an estimate of the solution near the blow-up time.

The motivation for considering the Cauchy problem for system (20.1.1), (20.1.2) it is a degenerate partial differential equation and therefore it is a source for the emergence of new nonlinear effects such as the finite velocity of perturbation propagation, the spatial localization of bounded and blow up solutions, the occurrence of which was first established in the work [1] for case $n = q_i = p_i = 0,\ k = 1,\ p = 2$. Nowadays the problem of the numerical modeling considered problem not studied enough.

20.2 Construction of the Self-similar System Equation

The study of various properties of solutions to system (20.1.1) is a difficult problem, even for a particular case of system (20.1.1) [1, 5–7]. In works [1, 5–7] in the case $n = 0,\ p = 2$ for the other nonlinear system (20.1.1), (20.1.2) were shown affectivity of the self-similar approach for studying of the different properties of the solutions of the problem (20.1.1) and (20.1.2). Below one manner—the method of nonlinear splitting for construction of self-similar. This manner are facilitated by more simple way investigate of qualitative properties of solutions of the problem (20.1.1) and (20.1.2). For the construction of self-similar systems, a nonlinear splitting algorithm is proposed [6]. The qualitative properties of solution considered problem based on self-similar, and approximately self-similar approach. Therefore, in order to reduce the system of equations (20.1.1) to a self-similar, approximately form, we first solve the system of ordinary differential equations

$$\frac{d\overline{u}}{dt} = -\gamma(t)\overline{u}^{p_1}\overline{v}^{q_1}, \quad \frac{d\overline{v}}{dt} = -\gamma(t)\overline{u}^{p_2}\overline{v}^{q_2} \tag{20.2.1}$$

which have a solution of the form

$$\overline{u}(t) = \left[T + \int_0^t \gamma(y)\,dy\right]^{-\alpha_1}, \quad \overline{v}(t) = \left[T + \int_0^t \gamma(y)\,dy\right]^{-\alpha_2},$$

where

$$\alpha_1 = \frac{(q_2+1) - q_1}{(p_1-1)(q_2-1) - p_2 q_1}, \quad \alpha_2 = \frac{(p_1+1) - p_2}{(p_1-1)(q_2-1) - p_2 q_1},$$

Then, in order to construct a self-similar and approximately self-similar system for system (20.1.1), we will seek solutions to the system of equations (20.1.1) in the form

$$u\left(t,x\right)=\overline{u}\left(t\right)w\left(\tau\left(t\right),\varphi\left(|x|\right)\right),\quad v\left(t,x\right)=\overline{v}\left(t\right)z\left(\tau\left(t\right),\varphi\left(|x|\right)\right) \tag{20.2.2}$$

where

$$\tau\left(t\right)=\int_{0}^{t}\overline{u}^{k(p-2)+m_1+l_1-2}\left(y\right)dy=\int_{0}^{t}\overline{v}^{k(p-2)+m_2+l_2-2}\left(y\right)dy$$

$$\varphi\left(|x|\right)=\frac{|x|^{g_1}}{g_1},\quad g_1=\frac{p}{p-(n+l)},\quad (n+l)<p$$

Substituting (20.2.2) to the system (20.1.1) it reduced to the following system of equations

$$\begin{array}{l}\frac{\partial w}{\partial\tau}=\varphi^{1-s}\frac{\partial}{\partial\varphi}\left(\varphi^{s-1}w^{m_1-1}\left|\frac{\partial w}{\partial\varphi}^{k}\right|^{p-2}\frac{\partial w}{\partial\varphi}^{l_1}\right)\\ \quad+\bar{u}^{p_1-k(p-2)-m_1-l_1+1}\bar{v}^{q_1}\left(w+\varepsilon w^{p_1}z^{q_1}\right)\\ \frac{\partial z}{\partial\tau}=\varphi^{1-s}\frac{\partial}{\partial r}\left(\varphi^{s-1}z^{m_2-1}\left|\frac{\partial z}{\partial\varphi}^{k}\right|^{p-2}\frac{\partial z}{\partial\varphi}^{l_2}\right)\\ \quad+\bar{u}^{p_2}\bar{v}^{q_2-k(p-2)-m_2-l_2+1}\left(z+\varepsilon z^{q_2}w^{p_2}\right)\end{array} \tag{20.2.3}$$

where $s=\frac{p(N-l)}{p-(n+l)},\ n+l<p$

It is easy to see that system (20.2.3) has the following an approximately self-similar solution of the form

$$w\left(\tau\left(t\right),\varphi\left(x\right)\right)=f_1\left(\xi\right),\quad z\left(\tau\left(t\right),\psi\left(x\right)\right)=f_2\left(\xi\right), \tag{20.2.4}$$

where $\varphi\left(x\right)=\frac{p}{p-(n+l)}|x|^{\frac{p-(n+l)}{p}}$, $\xi=\varphi\left(x\right)\tau^{-1/p}$ and the functions $f_1\left(\xi\right),\ f_2\left(\xi\right)$ satisfy to the approximately self-similar system

$$\begin{array}{l}L_1(s,p,m_1)f_1+\tau(t)\bar{u}^{p_1-k(p-2)-m_1-l_1+1}\bar{v}^{q_1}\alpha_1\left(f_1+{f_1}^{p_1}{f_2}^{q_1}\right)=0,\\ L_2(s,p,m_2)f_2+\tau(t)\bar{u}^{p_2}\bar{v}^{q_2-k(p-2)-m_2-l_2+1}\alpha_2\left(f_2+{f_2}^{q_2}{f_1}^{p_2}\right)=0,\end{array} \tag{20.2.5}$$

where

$$\begin{array}{l}L_1(s,p,m_1)f_1=\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}{f_1}^{m_1-1}\left|\frac{df_1}{d\xi}^{k}\right|^{p-2}\frac{df_1}{d\xi}^{l_1}\right)+\frac{\xi}{p}\frac{df_1}{d\xi}\\ L_2(s,p,m_2)f_2=\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}{f_2}^{m_2-1}\left|\frac{df_2}{d\xi}^{k}\right|^{p-2}\frac{d{f_2}^{l_2}}{d\xi}\right)+\frac{\xi}{p}\frac{df_2}{d\xi}.\end{array} \tag{20.2.6}$$

In particular if $\gamma(t) = 1$ then

$$\begin{aligned} \tau(t)\bar{u}^{p_1-k(p-2)-m_1-l_1+1}\bar{v}^{q_1} &= c_1 = \tfrac{\alpha_1}{1-(k(p-2)+m_1+l_1-1)\alpha_1}, \\ \tau(t)\bar{u}^{p_2}\bar{v}^{q_2-k(p-2)-m_2-l_2+1} &= c_2 = \tfrac{\alpha_2}{1-(k(p-2)+m_2+l_2-1)\alpha_2}. \end{aligned} \tag{20.2.7}$$

Therefore, we have the following the self-similar system

$$\begin{aligned} &L_1(f_1, f_2) \equiv \xi^{1-s}\tfrac{d}{d\xi}\left(\xi^{s-1}{f_1}^{m_1-1}\left|\tfrac{df_1}{d\xi}^k\right|^{p-2}\tfrac{d{f_1}^{l_1}}{d\xi}\right) + \tfrac{\xi}{p}\tfrac{df_1}{d\xi} + \\ &\tfrac{\alpha_1}{1-(k(p-2)+m_1+l_1-1)\alpha_1}\left(f_1 + \varepsilon {f_1}^{p_1}{f_2}^{q_1}\right) = 0, \\ &L_2(f_1, f_2) \equiv \xi^{1-s}\tfrac{d}{d\xi}\left(\xi^{s-1}{f_2}^{m_2-1}\left|\tfrac{df_2}{d\xi}^k\right|^{p-2}\tfrac{d{f_2}^{l_2}}{d\xi}\right) + \tfrac{\xi}{p}\tfrac{df_2}{d\xi} + \\ &\tfrac{\alpha_2}{1-(k(p-2)+m_2+l_2-1)\alpha_2}\left(f_2 + \varepsilon {f_2}^{q_2}{f_1}^{p_2}\right) = 0. \end{aligned} \tag{20.2.8}$$

$$(k(p-2) + m_1 + l_1 - 1)\alpha_1 = (k(p-2) + m_2 + l_2 - 1)\alpha_2$$

Theorem 20.2.1 *Let* $k(p-2) + m_i + l_i - 1 > 0$, $p > n + l$, $\tau(t)\bar{u}^{p_i-k(p-2)-m_i-l_i+1}\bar{v}^{q_i}\alpha_i < \frac{s}{p}$, $i = 1, 2$ *for* $\forall t > 0$, $u(0, x) \le u_+(0, x)$, $v(0, x) \le v_+(0, x)$, $x \in R^N$. *Then problem* (20.1.1), (20.1.2) *for small data is global solvability and the estimate* $u(t, x) \le u_+(t, x)$, $v(t, x) \le v_+(t, x)$ *for solution and free boundary* $|x| \le \frac{p-n-l}{p} a^{\frac{p-1}{p}} [\tau(t)]^{\frac{1}{p-n-l}}$ *in* $Q = \left\{(t, x) : \ t > 0, \ x \in R^N\right\}$ *holds, where*

$$u_+(t, x) = \left(T + \int_0^t \gamma(y)\, dy\right)^{-\alpha_1} f_1(\xi), \quad v(t, x) = \left(T + \int_0^t \gamma(y)\, dy\right)^{-\alpha_2} f_2(\xi),$$
$$f_i(\xi) = (a - \xi^\gamma)_+{}^{n_i}, \quad n_i = \tfrac{p-1}{k(p-2)+m_i+l_i-2}, \quad i = 1, 2$$

Proof Proof of the theorem based on comparison principle. For comparison function we take $u_+(t, x)$, $v_+(t, x)$. It is easy to check that

$$L_1(u_+, v_+) \le 0, \quad L_2(u_+, v_+) \le 0$$

in

$$D = \left\{(t, x) : \ t > 0, \ |x| \le \frac{p-n-l}{p} a^{\frac{p-1}{p}} [\tau(t)]^{\frac{1}{p-n-l}}\right\}.$$

Then according comparison principle we have $u(t, x) \le u_+(t, x)$, $v(t, x) \le v_+(t, x)$in $Q = \left\{(t, x) : \ t > 0, \ x \in R^N\right\}$. Proof of Theorem 1 is completed. □

The case $p = n + l$ we will call singular case [8, 9]. In this case we have the following

Theorem 20.2.2 *Let* $k(p-2)+m_i+l_i-1 > 0,\ p = n+l, \tau(t)\bar{u}^{p_i-k(p-2)-m_i-l_i+1}\bar{v}^{q_i}$ $\alpha_i < \frac{s}{p},\ i = 1, 2$ *for* $\forall t > 0,\ u(0, x) \le u_+(0, x),\ \ v(0, x) \le v_+(0, x),\ x \in R^N \backslash \{0\}$. *Then problem* (20.1.1), (20.1.2) *for small data is global solvable and the estimate* $u(t, x) \le u_+(t, x),\ v(t, x) \le v_+(t, x)$ *for solution in* $Q = \left\{(t, x) : t > 0,\ x \in R^N \backslash \{0\}\right\}$ *for free boundary* $|x| > \exp\left(a^{\frac{p-1}{p}}[\tau(t)]^{\frac{1}{p}}\right)$ *holds,*
where

$$u_+(t, x) = \left(T + \int\limits_0^t \gamma(y)\,dy\right)^{-\alpha_1} f_1(\xi), \quad v(t, x) = \left(T + \int\limits_0^t \gamma(y)\,dy\right)^{-\alpha_2} f_2(\xi),$$
$$f_i(\xi) = (a - \xi^{\gamma})_+{}^{n_i}, \quad n_i = \frac{p-1}{k(p-2)+m_i+l_i-2}, \quad i = 1, 2$$

20.3 Asymptotic of Self-similar Solution

Consider self-similar solutions to system (20.2.8) satisfying the following boundary conditions:

$$f_1(0) = c_0 > 0,\ f_1(b_1) = 0,\ b_1 < \infty,\ f_2(0) = c_0 > 0,\ f_2(b_2) = 0,\ b_2 < \infty, \tag{20.3.1}$$

$$f_1(0) = c_0 > 0,\ f_1(\infty) = 0,\ f_2(0) = c_0 > 0,\ f_2(\infty) = 0. \tag{20.3.2}$$

Let us study the asymptotics of solutions to problem (20.2.8), (20.3.1).

Theorem 20.3.1 *Let* $m_i + l_i + k(p-2) - 1 > 0,\ \ i = 1, 2$. *Then the solution* $f_1(\xi),\ f_2(\xi)$ *of the system* (20.2.8) *for* $\xi \to a^{\frac{1}{\gamma}}$ *has asymptotics*

$$\begin{aligned} f_1(\xi) &= c_1 \bar{f}_1(\xi)\,(1 + o(1)) \\ f_2(\xi) &= c_2 \bar{f}_2(\xi)\,(1 + o(1)) \end{aligned} \tag{20.3.3}$$

where coefficients c_1, c_2 *satisfy the system of algebraic equations*

$$\begin{aligned} (n_1(k(p-2) + m_1 + l_1) - p + 1)\,\gamma n_1 |k n_1|^{p-2} c_1{}^{m_1+k(p-2)+l_1} + a c_1{}^{p_1} c_2{}^{q_1} &= 0, \\ (n_2(k(p-2) + m_2 + l_2) - p + 1)\,\gamma n_2 |k n_2|^{p-2} c_2{}^{m_2+k(p-2)+l_2} + a c_1{}^{p_2} c_2{}^{q_2} &= 0. \end{aligned} \tag{20.3.4}$$

Proof To prove it, we will seek a solution to system (20.2.8) in the form

$$\begin{aligned}
&f_1(\xi) = \overline{f_1}(\xi)\, w(\tau), \quad \overline{f_1}(\xi) = (a - \xi^{\gamma})_+{}^{n_1}, \ \gamma = \tfrac{p}{p-1}, \ \tau = -\ln(a - \xi^{\gamma}),\\
&f_2(\xi) = \overline{f_2}(\xi) z(\tau), \quad \overline{f_2}(\xi) = (a - \xi^{\gamma})_+{}^{n_2}, \ \ \xi = \varphi(x)\, \tau(t)^{-1/p},\\
&\varphi(x) = \tfrac{p}{p-n-l} |x|^{\frac{p-n-l}{p}},
\end{aligned} \tag{20.3.5}$$

where

$$n_1 = \tfrac{p-1}{k(p-2)+m_1+l_1-2}, \quad n_2 = \tfrac{p-1}{k(p-2)+m_2+l_2-2}, \tag{20.3.6}$$

Substituting (20.3.5) into (20.2.8) after simple calculations we obtain the following system

$$\begin{aligned}
&\left[\left(s\tfrac{e^{-\tau}}{a-e^{-\tau}} - n_1\left(k(p-2) + m_1 + l_1\right) - p + 1\right)\gamma L_1(w) + \gamma \tfrac{d}{d\tau} L_1(w)\right] +\\
&\tfrac{\gamma e^{-\tau}\left(n_1 - \left(n_1(k(p-2)+m_1+l_1) - p + 1\right)\right)}{p} \tfrac{dw}{d\tau} + \left(a - e^{-\tau}\right) w^{p_1} z^{q_1} = 0,\\
&\left[\left(s\tfrac{e^{-\tau}}{a-e^{-\tau}} - n_2\left(k(p-2) + m_2 + l_2\right) - p + 1\right)\gamma L_2(z) + \gamma \tfrac{d}{d\tau} L_2(z)\right] +\\
&\tfrac{\gamma e^{-\tau}\left(n_2 - \left(n_1(k(p-2)+m_1+l_1) - p + 1\right)\right)}{p} \tfrac{dw}{d\tau} + \left(a - e^{-\tau}\right) w^{p_2} z^{q_2} = 0,
\end{aligned} \tag{20.3.7}$$

The analysis of the solution of the last system shows that $w \to c_1, \quad z \to c_2$ for $\tau \to \infty$ where the constants c_1, c_2 are solutions of the algebraic system of

$$\begin{aligned}
&\left(n_1\left(k(p-2) + m_1 + l_1\right) - p + 1\right)\gamma n_1 |kn_1|^{p-2} c_1{}^{m_1+k(p-2)+l_1} + ac_1{}^{p_1} c_2{}^{q_1} = 0,\\
&\left(n_2\left(k(p-2) + m_2 + l_2\right) - p + 1\right)\gamma n_2 |kn_2|^{p-2} c_2{}^{m_2+k(p-2)+l_2} + ac_1{}^{p_2} c_2{}^{q_2} = 0.
\end{aligned} \tag{20.3.8}$$

Theorem 2 is proved. □

Theorem 20.3.2 *Let* $m_i + l_i + k(p-2) - 1 < 0, \quad i = 1, 2$. *Then the regular solution* $f_1(\xi), \quad f_2(\xi)$ *of the problem* (20.2.8), (20.3.2) *for* $\xi \to \infty$ *has asymptotics*

$$f_1(\xi) = c_1 \bar{f}_1(\xi)(1 + o(1)), \ f_1(\xi) = c_2 \bar{f}_2(\xi)(1 + o(1)) \tag{20.3.9}$$

where the coefficients c_1, c_2*satisfy the system of algebraic equations*

$$\begin{aligned}
&\left(s + \left(n_1(k(p-2) + m_1) - p + 1\right)\right)\gamma n_1 |kn_1|^{p-2} c_1{}^{m_1+k(p-2)} + \varepsilon c_1{}^{p_1} c_2{}^{q_1} = 0,\\
&\qquad \varepsilon = \pm 1,\\
&\left(s + \left(n_2(k(p-2) + m_2) - p + 1\right)\right)\gamma n_2 |kn_2|^{p-2} c_2{}^{m_2+k(p-2)} + \varepsilon c_1{}^{p_2} c_2{}^{q_2} = 0
\end{aligned} \tag{20.3.10}$$

Proof Proof of Theorem 20.3.2 is similar to the proof of the Theorem 20.3.1. □

Theorem 20.3.3 *Let in* (20.2.8) $p = n + l, m_i + l_i + k(p-2) - 1 = 0$. *Then solution of problem* (20.2.8), (20.3.2) *at* $\xi \to \infty$ *have the following asymptotics*

$$\begin{aligned} f_1(\xi) &= c_3 \exp(-b\xi^{\gamma})(1+o(1)), \quad \gamma = \tfrac{p}{p-1} \\ f_2(\xi) &= c_4 \exp(-b\xi^{\gamma})(1+o(1)), \quad \xi = \varphi(x)\,\tau(t)^{-1/p},\ \varphi(x) = \ln|x| \end{aligned} \tag{20.3.11}$$

where $b = \left(\frac{m_i^{p_i}}{k^{p-2}}\right)^{\frac{1}{p-p_i}}$.

Proof Proof of Theorem 20.3.2 is similar to the proof of the Theorem 20.3.1. □

20.4 Results of the Numerical Experiments of the Solutions

The numerical solution of problem (20.1.1), (20.1.2) is a difficult task due to degeneration of the considered problem and numerical results depend on the manner of the linearization and method in the numerical solution of a nonlinear system (20.1.1) by iterative methods. In the work for numerical solution used modification of the sweep method suggested by Samarskii A., Sobol I. [5] For numerical solution linearized system used iterative processes based on the Picard and Newton method. For the numerical solution in two dimensional case of this problem, the method of variable directions is used, with the Peaceman-Rachford scheme of the following form

$$\begin{cases} \frac{y_i^{j+1}-y_i^j}{\tau} = \frac{\left(y_{i+1}^j\right)^n}{h^2}\Big[|x_{i+1}|^n a_{i+1}\left(y^j\right)\left(y_{i+1}^{j+1} - y_i^{j+1}\right) - a_i\left(y^j\right)\left(y_i^{j+1} - y_{i-1}^{j+1}\right) \\ \qquad\qquad -\varepsilon d_i\left(y^{j+1}, z^{j+1}\right)\Big], \\ \frac{z_i^{j+1}-z_i^j}{\tau} = \frac{\left(z_{i+1}^j\right)^n}{h^2}\Big[|x_{i+1}|^n b_{i+1}\left(z^j\right)\left(z_{i+1}^{j+1} - y_i^{j+1}\right) - b_i\left(z^j\right)\left(z_i^{j+1} - z_{i-1}^{j+1}\right) \\ \qquad\qquad -\varepsilon d_i\left(z^{j+1}, y^{j+1}\right)\Big], \\ i = 1, 2, \ldots, n-1;\ \mathrm{j} = 0, 1, \ldots, m_1 - 1 \\ y_i^0 = u_0(x_i),\ \mathrm{z}_i^0 = v_0(x_i),\ i = 0, 1, \ldots, n_1 \\ y_0^j = \phi_1\left(\tau_j\right),\ \mathrm{z}_0^j = \varphi_1\left(\tau_j\right)\ j = 1, 2, \ldots, m_1 \\ y_n^j = \phi_2\left(\tau_j\right),\ \mathrm{z}_n^j = \varphi_2\left(\tau_j\right),\ j = 1, 2, \ldots, m_1 \end{cases}.$$

where

$$d_i\left(y^{j+1}, z^{j+1}\right) = \left(\overset{r}{y}_i\right)^{p_i}\left(\overset{r}{z}_i\right)^{q_i} - \text{Picard method},$$

$$d_i\left(y^{j+1}, z^{j+1}\right) = \left[y_i^j + p_1\left(y_i^j\right)^{p_1-1}\left(y_i^{j+1} - y_i^j\right)\right]\left(z_i^j\right)^{q_1}.$$

$$d_i\left(z^{j+1}, y^{j+1}\right) = \left[z_i^j + p_2\left(z_i^j\right)^{p_2-1}\left(z_i^{j+1} - z_i^j\right)\right]\left(y_i^j\right)^{q_2} - \text{ Newton's method.}$$

$$a_{i+1}(y^{j+1}) = \frac{1}{2}\left[\begin{array}{l} (y_i^{j+1})^{m_1-1}\left|\frac{(y_{i+1}^{j+1})^k-(y_i^{j+1})^k}{h}\right|^{p-2} + \\ +(y_{i+1}^{j+1})^{m_1-1}\left|\frac{(y_i^{j+1})^k-(y_{i-1}^{j+1})^k}{h}\right|^{p-2} \end{array}\right],$$

$$b_{i+1}(z^{j+1}) = \frac{1}{2}\left[\begin{array}{l} (z_i^{j+1})^{m_2-1}\left|\frac{(z_{i+1}^{j+1})^k-(z_i^{j+1})^k}{h}\right|^{p-2} + \\ +(z_{i+1}^{j+1})^{m_2-1}\left|\frac{(z_i^{j+1})^k-(z_{i-1}^{j+1})^k}{h}\right|^{p-2} \end{array}\right],$$

$$a_i(y^{j+1}) = \frac{1}{2}\left[\begin{array}{l} (y_{i-1}^{j+1})^{m_1-1}\left|\frac{(y_i^{j+1})^k-(y_{i-1}^{j+1})^k}{h}\right|^{p-2} + \\ +(y_i^{j+1})^{m_1-1}\left|\frac{(y_{i-1}^{j+1})^k-(y_{i-2}^{j+1})^k}{h}\right|^{p-2} \end{array}\right],$$

$$b_i(y^{j+1}) = \frac{1}{2}\left[\begin{array}{l} (z_{i-1}^{j+1})^{m_2-1}\left|\frac{(z_i^{j+1})^k-(z_{i-1}^{j+1})^k}{h}\right|^{p-2} + \\ +(z_i^{j+1})^{m_2-1}\left|\frac{(z_{i-1}^{j+1})^k-(z_{i-2}^{j+1})^k}{h}\right|^{p-2} \end{array}\right],$$

where $r = 0, 1, 2, \ldots$

The differential circuit is linear in relation to the functions $\overset{(r+1}{y}{}^{j+1}$, $\overset{(r+1)}{z}{}^{j+1}$, $\overset{(0}{y}{}^{j+1} = y^j$, $\overset{(0)}{z}{}^{j+1} = z^j$ from the previous time step is taken as the initial iteration.

The results of computational experiments show that both iterative methods are effective for solving nonlinear problems due to appropriate choosing of the initial approximation. As expected Newton's method requires fewer iterations than Picard's method to achieve necessary accuracy (see table below):

Numerical parameters						Number of iterations	
p	k	n	m_i	q_i	p_i	Newton	Picard
6.9	3.5	3	3.2	2.3	2.1	3	3
4.7	2.5	3	1.7	2.2	2.9	3	4
2.9	4.1	1	2	2	1.5	3	4
3	3	1	2	4	5	2	3
3	1	3	2	2	1	2	2

Figures 20.1, 20.2, 20.3, and 20.4 show that results of the numerical experiment gives the effect of a finite speed of a perturbation of solution, and localization of solution depending on value numerical parameters. The computational experiment were carried out for a slowly and a fast diffusion cases.

1. Slowly diffusive case. For initial approximation $(m_i + l_i + k(p-2) - 1 > 0,\ i = 1, 2)$ used the following functions:

$$
\begin{aligned}
&\mathrm{u}_0\,(t,x) = \overline{u}(t)\overline{f}_1\,(\xi)\,,\quad v_0\,(t,x) = \overline{v}(t)\overline{f}_2\,(\xi)\,,\\
&\overline{f}_1(\xi) = (a - \xi^{\gamma})^{n_1},\ \overline{f}_2(\xi) = (a - \xi^{\gamma})^{n_2},\ \gamma = p/(p-1),\ \ \xi = \varphi\,(x)\,\tau^{-1/p},\\
&\varphi\,(x) = \tfrac{p}{p-(n+l)}|x|^{\frac{p-(n+l)}{p}},\\
&\overline{u}(t) = (T+t)^{-\alpha_1},\quad \overline{v}(t) = (T+t)^{-\alpha_2}.
\end{aligned}
$$

2. Fast diffusive case. In this cases an initial approximation $(m_i + l_i + k(p-2) - 1 < 0,\ \ i = 1, 2)$ following functions are used:

$$
\begin{aligned}
&\mathrm{u}_0\,(t,x) = \overline{u}(t)\overline{f}_1\,(\xi)\,,\quad v_0\,(t,x) = \overline{v}(t)\overline{f}_2\,(\xi)\,,\\
&\overline{f}_1(\xi) = (a + \xi^{\gamma})^{n_1},\ \overline{f}_2(\xi) = (a + \xi^{\gamma})^{n_2},\ \gamma = p/(p-1),\ \ \xi = \varphi\,(x)\,\tau^{-1/p},\\
&\varphi\,(x) = \tfrac{p}{p-(n+l)}|x|^{\frac{p-(n+l)}{p}},\\
&\overline{u}(t) = (T+t)^{-\alpha_1},\quad \overline{v}(t) = (T+t)^{-\alpha_2},
\end{aligned}
$$

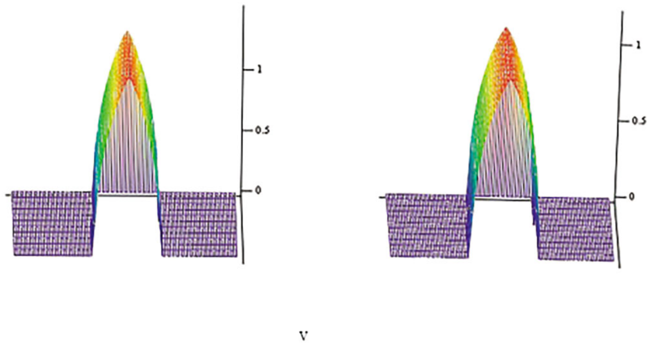

Fig. 20.1 The results of the numerical experiment for slow diffusive case-1

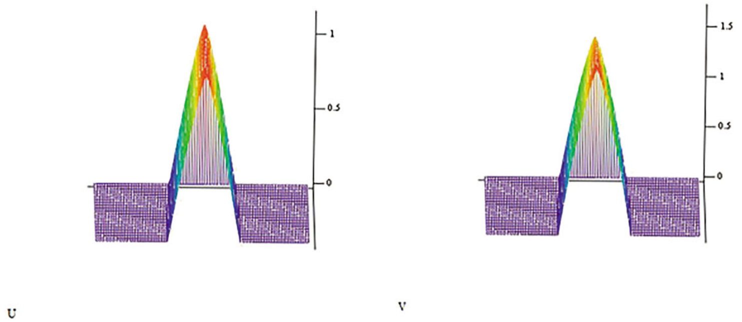

Fig. 20.2 The results of the numerical experiment for slow diffusive case-2

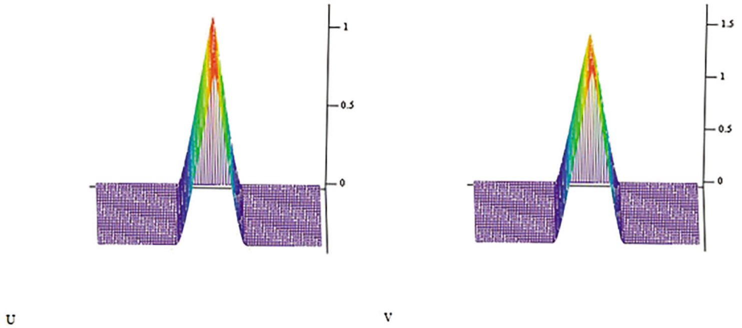

Fig. 20.3 The results of the numerical experiment for fast diffusive case-3

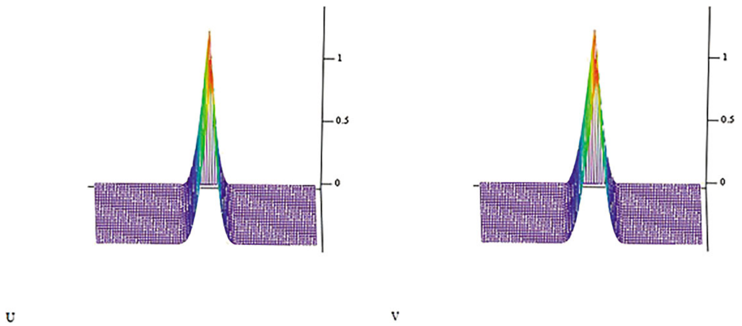

Fig. 20.4 The results of the numerical experiment for fast diffusive case-4

20.5 Conclusion

In this paper, the qualitative properties of the problem Cauchy for double nonlinear system with variable density and nonlinear source or absorption based on self-similar analysis of solutions, the influence of variable density, source or absorption to evolution of studied processes established. The asymptotic behavior of self-similar solutions depending on the value of the numerical parameters of system (20.1.1) investigated.. The problem of choosing initial approximations for the numerical analysis of solutions of the considered problem is solved. It is shown that the coefficient at the principal term of the asymptotics of the solution satisfies to some system of nonlinear algebraic equations. For numerical solution the iterative processes are built on the basis of the Picard, Newton methods. The results of computational experiments show that both iterative methods are effective for numerical solution considered double nonlinear problems and lead to new nonlinear effects due to suggested appropriate an initial approximation the solutions for the numerical solution. The results of computational experiments show that both iterative method and computational scheme and sweep iterative methods are effective for solving considered nonlinear problem and lead to nonlinear effects if solutions of self-similar equations constructed by the nonlinear splitting and the standard equation methods are used as the initial approximation.

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References

1. Samarskii, A., Kurdyumov, S., Galaktionov, V., Mikhailov, A.: Blow Up in Quasilinear Parabolic Equations. Nauka, Moscow (1986)
2. Aripov, M.: Asymptotics of the solution of the non-Newton politropic filtration equation. ZAMM-Berlin Z. Angew. Math. Mech. **80**, 767–768 (2000)
3. Martynenko A., Tedeev, A.: On the behavior of solutions to the cauchy problem for a degenerate parabolic equation with inhomogeneous density and a source. Comput. Math. Math. Phys. **7**, 1145–1160 (2008)
4. Fujita, H.: On the blowing up of solutions to the cauchy problem for ut = du + u1a. J. Fac. Sci. Univ. Tokyo Sect. **13**, 109–124 (1966)
5. Samarskii, A., Sobol, I.: Examples of the numerical calculation of temperature waves. Comput. Math. and Math. Phys. **4**, 702–719 (1963)
6. Aripov, M.: Approximate self-similar approach for solving of quasilinear parabolic equation. In: Experimentation, Modeling and Computation in Flow, Turbulence and Combustion, vol. 6, pp. 19–26. Wiley (1997)
7. Kersner, R., Reyes, J., Tesei, A.: One a class of nonlinear parabolic equations with variable density and absorption. Adv. Differential Equations **7**, 155–176 (2002)
8. Aripov, M., Sadullaeva, S.: Computer Modeling of Nonlinear Diffusion Processes, 687 pp. Tashkent University (2020)
9. Nigmanova, D.: To numerical investigation of non-linear reaction-diffusion processes with variable density and source. Probl. Comput. Appl. Math. **3**(4), 36–50 (2023)

Chapter 21
An Extensive Simulation Study for Evaluation of Penalized Variable Selection Methods in Logistic Regression Model with High Dimensional Data

Nuriye Sancar and Ayad Bacar

Abstract Variable selection, as a category of supervised methods, is a procedure in statistics that involves selecting a subset of important variables from a larger set of variables. The variable selection process in high dimensional data is quite significant to avoid overfitting and produces meaningful results from the model. Lasso, Elastic Net, Adaptive Lasso, and Adaptive Elastic Net, known as penalized methods, are frequently used methods for variable selection to reduce dimensionality in the logistic regression model with high dimensional data. This research aims to examine and compare the performances of these penalized methods in the variable selection process in logistic regression under different scenarios through an extensive simulation study in high- dimensional data.

21.1 Introduction

Today, in an era of rapidly evolving data science, researchers are faced with datasets of increasing size and complexity [1]. In this context, high-dimensional data analysis has become a hot topic in data science. In high-dimensional data, unlike low-dimensional data, the number of variables (p) is greater than the number of samples (n). Researchers frequently deal with such data in genetic, biological, financial, and many other fields. Although high-dimensional datasets provide valuable information because they contain greater detail and complexity, they can also complicate analysis processes and lead to overfitting of models. When working with high-dimensional data sets, the variable selection process is quite important so that the model avoids overfitting and produces meaningful results from the model [2]. Variable selection, as a category of supervised methods, is a procedure in statistics

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that involves selecting a subset of important variables from a larger set of variables. This process is used to both improve the performance of the model and reduce the computational cost.

The logistic regression model is a probability model that estimates the probability that the dependent variable belongs to one of two categories. Logistic regression is frequently used by researchers to achieve successful results in high-dimensional data sets, especially in classification problems. In logistic regression with high-dimensional data, variable selection is essential to improve model interpretability and prediction performance, and to reduce overfitting [3]. In data science, there are generally three types of variable selection methods: filter, wrapper, and embedded. Each of these methods has different advantages and characteristics in terms of determining variables and increasing model performance [1]. In filter approaches that work independently of the model, a value is calculated for each variable in the data set through an evaluation function, and the variables with the highest values among these calculated values are selected for the best variable subset. These methods are fast but ignore how variables interact with the final model. Wrapper methods, another variable selection method, evaluate variables according to the performance of the model using the model to be trained. Wrapper methods are generally accurate and flexible, but are computationally costly and run the risk of overfitting.

On the other hand, Embedded methods carry out the classification and variable selection processes simultaneously since they contain both the classification algorithm and the variable selection algorithm in their structure. Although these techniques are successful and efficient, a thorough understanding of the model and its characteristics is necessary. The Embedded approach has a much lower tendency to overfit than other approaches. Since embedded methods perform variable selection during model training, they can be considered more advantageous than other methods thanks to their ability to handle many variables at the same time [4]. It can also handle many variables simultaneously and increase generalization ability by offering flexibility to control the complexity of the model by adjusting the penalizing parameters. Decision tree-based approaches (such as random forest [5], and gradient boosting [6]) and variable selection utilizing penalization methods (such as Lasso [7], Adaptive Lasso [8], Elastic Net [9], and Adaptive Elastic Net [10]) are some types of embedded approaches. Penalization methods have some advantages over decision tree based methods such as interpretation of model coefficients, simple parameter settings, and more tendency to result in sparse solutions Although Random Forest is that is resistance to overfitting in high-dimensional datasets, Random Forest is not generally preferred as a variable selection method in high dimensional data because of memory and computational limitations [11]. Also, The Random Forest algorithm's performance declines when redundant variables exist in the dataset [12]. On the other hand, the advantages of Gradient Boosting include strong predictive performance and resistance to outliers. However it may face overfitting in high-dimensional datasets [13].

Lasso, Elastic Net, Adaptive Lasso, and Adaptive Elastic Net, known as penalized methods, are frequently used among these embedded methods for variable selection to reduce dimensionality in high dimensional data. These penalized methods from embedded methods are faster and more accurate than other methods [14]. This study aims to examine and compare the performances of these penalized methods in the variable selection process in logistic regression under different scenarios through an extensive simulation study in high-dimensional data.

21.2 Methodology

A binary logistic regression (BLR) is used to predict the likelihood of a binary event through a logit link function. The binary logistic regression model is defined in Eq. (21.1):

$$P(y_i = 1|x_i, \beta) = \pi(x_i) = \frac{1}{1 + e^{-x_i\beta}}. \tag{21.1}$$

In the context of a logistic regression model, the dependent variable, denoted as y_i follows a Bernoulli distribution represented as $y_i \sim B(\pi_i)$. x_i is row i of the independent variable matrix with n observations and p independent variables, and $\beta_{(p+1)\times 1}$ is the column vector of model coefficients. The maximum likelihood (ML) method, which aims to maximize log likelihood defined in Eq. (21.2), is used to estimate model coefficient β

$$L(\beta) = \ln l(\beta) = \sum_{i=1}^{n} y_i x_i'\beta - \ln\left[1 + e^{x_i'\beta}\right]. \tag{21.2}$$

BLR with high-dimensional datasets has some challenges in the modeling process, as the theoretical structure of the BLR prevents parameter estimation through the maximization of log likelihood when dealing with such data. The model becomes ineffective when the data is high-dimensional because maximizing the likelihood produces abnormally large regression coefficients in the model, which cause overfitting and erroneous estimations. On the other hand, due to the high dimensionality, the independent variables are frequently strongly correlated, leading to a multicollinearity problem in the independent variable matrix. The multicollinearity issue warrants consideration because it results in high standard errors of the parameters and leads to erroneous results because of unstable parameter estimations. For the BLR with high dimensional data, numerous penalized approaches have been developed to address the multicollinearity issue and for variable selection. LASSO, Adaptive LASSO, Elastic-net, and Adaptive Elastic-net are frequently used penalized methods in BLR with high-dimensional data.

The least absolute shrinkage and selection operator (LASSO), also known as L1-penalization, introduced by Tibshirani in 1996 [7], aims to reduce parameters to zero to prevent overfitting. The procedure for calculating the LASSO penalty involves multiplying the coefficients' absolute value by the penalization parameter λ, added to $L(\beta)$. LASSO estimator in logistic regression is obtained mathematically as follows

$$\hat{\beta}_{\mathrm{i(LASSO)}} = \arg\min_{\beta}\left[-L(\beta) + \lambda \sum_{i=1}^{p} |\beta_i|\right]. \tag{21.3}$$

This prevents overfitting by reducing the parameters to zero. Upon $\lambda = 0$, it simplifies to the ML estimator. If, however, $\lambda \to \infty$, then all the independent variables become 0 due to the penalization term. Generally, the penalization parameter, λ is selected by cross-validation method. While the LASSO has certain benefits, there are drawbacks when it comes to computationally handling high-dimensional data. First, because of convex optimization restrictions, the LASSO can only choose a maximum of n variables when $p >> n$. Second, it has trouble managing the effects of grouping, especially when there are strong pairwise correlations between a set of independent variables, leading to the selection of only one variable without considering which one. Lastly, Fan and Li (2001) have pointed out that there is no oracle characteristics (consistency of feature selection) in the LASSO method [15]. Adaptive LASSO (AdaLASSO) estimator was proposed by Zou [8]. AdaLASSO estimator for the logistic model is obtained as the following Eq. 21.4:

$$\hat{\beta}_{\mathrm{i(adaLASSO)}} = \arg\min_{\beta}\left[-L(\beta) + \lambda \sum_{i=1}^{p} \frac{|\beta_i|}{|\beta_i'|}\right], \tag{21.4}$$

where $\hat{\beta}$ is an estimator of β, and the ridge estimator is usually utilized as an acceptable estimator of β. The existing studies have shown that AdaLASSO has oracle characteristics. If the oracle characteristics exist, the coefficients of the accurate model with 0 values are correctly estimated as zero, while the rest of the coefficients are calculated as if the actual model was known beforehand [10].

Zou and Hastie (2005) proposed the Elastic-net estimator for selecting variables that aim to address the limitations of LASSO [9]. Elastic net uses both ridge regression penalty to address the multicollinearity problem and the LASSO penalization in the variable selection feature in an attempt to combine the L1 and L2 penalizations. In other words, the Elastic net estimator is an estimator that aims to obtain an equilibrium among the LASSO and ridge estimators and is given by

$$\hat{\beta}_{\mathrm{i(Elastic\text{-}Net)}} = \arg\min_{\beta} -\frac{1}{n}L(\beta) + \lambda\left[\alpha \sum_{i=1}^{p} |\beta_i| + (1-\alpha)\frac{1}{2}\sum_{i=1}^{p} \beta_i^2]\right], \tag{21.5}$$

where $\lambda \geq 0$ and $0 \leq \alpha \leq 1$. If $\alpha = 1$, the Elastic-net estimator is reduced to LASSO estimator. If $\alpha = 0$, the Elastic-net estimator is reduced to the Ridge estimator, $\hat{\beta}_{i\,\mathrm{(Ridge)}} = \arg\min_{\beta}\left[-L(\beta) + \lambda \sum_{i=1}^{p} \beta_i^2\right]$. The elastic-net has an advantageous property called the grouping effect, in which a group of variables that are highly correlated have identical coefficients and are chosen simultaneously.

The Adaptive elastic-net (AdaElastic-Net) is proposed by Zou and Zhang [10]. It includes the strengths of quadratic penalization and adaptively weighted LASSO shrinkage which handles the grouping effect and oracle characteristics. Adaptive elastic-net estimator can be considered as the combination of the elastic-net and the adaptive LASSO and is given by

$$\hat{\beta}_{i\,(\text{AdaElastic-Net})} = \arg\min_{\beta} -\frac{1}{n} L(\beta) + \lambda \left[\alpha \sum_{i=1}^{p} w_i \, |\beta_i| + (1-\alpha) \frac{1}{2} \sum_{i=1}^{p} \beta_i^2 \right]. \tag{21.6}$$

Here, $\hat{w}_i = \left(\left| \hat{\beta}_{i\,(\text{Elastic-Net})} + \frac{1}{n} \right| \right)^{-\gamma}$ with $\gamma > 0$. $\gamma = 1$ was used for this research since the results in the existing studies show that the estimate is not considerably impacted by this parameter [16–18]. To determine the most practical combinations, λ and α are often selected by optimizing k-fold cross-validation using grid search across α and λ [19]. The main purpose of this study is to examine in detail the advantages and disadvantages of the frequently used penalized methods through different simulation scenarios in an extensive simulation study on high-dimensional data for logistic regression.

21.3 Simulation Study

Simulation analysis has been conducted to compare the variable selection performances of LASSO, Adaptive-LASSO, Elastic-net, and Adaptive Elastic-net from penalized methods in logistic regression under different scenarios including different degree of collinearity from low to strong and different dimensionality through an extensive simulation study in high-dimensional data.

The logistic regression coefficients are assigned constant values denoted as β. The actual model coefficients are taken as $\beta = (\underbrace{1.5}_{20}, \underbrace{0, \ldots, 0}_{p-20})$ where the number of active independent variables is 20. We have taken $n = 150$ and $p = \{200, 300\}$. The generation of the response variable (y_i) is associated with the probability defined as $\pi(x_i) = \frac{1}{1+e^{-x_i \beta}}$. The variable y_i takes values of 0 or 1, where a probability, $\pi(x_i)$ is greater than or equal to 0.50 corresponds to $y_i = 1$, and $\pi(x_i)$ is less than 0.50 corresponds to $y_i = 0$. The penalization parameters, λ and α for the methods are selected using a fivefold cross-validation approach through grid search.

This simulation study has taken into consideration 8 different settings as follows:

Setting I: Covariates in the independent variables matrix X are generated from the normal distribution with low correlation ($r = 0.30$) for $p = 200$.

Setting II: Similar to Setting I, with moderate correlation ($r = 0.60$) for $p = 200$.

Setting III: Similar to Setting I, with strong correlation ($r = 0.90$) for $p = 200$.
Setting IV: Similar to Setting I, but with $p = 300$.
Setting V: Similar to Setting II, but with $p = 300$.
Setting VI: Similar to Setting III, but with $p = 300$.
Setting VII: Covariates in the independent variables matrix X include a grouping effect.
Group A: $x_i = d_1 + \xi_i, d_1 \sim N(0, 1)$ for $i = 1, 2, \ldots, 5$.
Group B: $x_i = d_2 + \xi_i, d_2 \sim N(0, 1)$ for $i = 6, 7, \ldots, 10$.
Group C: $x_i = d_3 + \xi_i, d_3 \sim N(0, 1)$ for $i = 11, 12, \ldots, 15$.
Group D: $x_i = d_4 + \xi_i, d_4 \sim N(0, 1)$ for $i = 16, 17, \ldots, 20$.
x_i is distributed as independent and identical with $x_i \sim N(0, 1)$ for $i = 21, 22, \ldots, p$, $\xi_i \sim N(0, 0.01)$ for $i = 1, 2, \ldots, 20$.
Setting VIII: Similar to Setting VII, but with $p = 300$.

The covariates in settings VII and VIII are derived including the grouped variables with four identical groups, where the relationship among the identical group is as strong as 0.99.

21.3.1 Performance Assessment Criteria for the Methods

The effectiveness of the models in variable selection has been assessed by F1score, true negative (TN), false negative (FN), true positive (TP), true negative (TN), sum of square error for the model coefficients (SSE), and precision (P), and recall (R). F1-score, TN, FN, TP, and FP have been evaluated by the confusion matrix. In this matrix: True Negative (TN): Correctly identified zero coefficients, True Positive (TP): Correctly identified non-zero coefficients, False Positive (FP): Incorrectly identifying zero coefficients as non-zero, False Negative (FN): Incorrectly identifying non-zero coefficients as zero. Precision (P), Recall (R) (also called sensitivity), and F1-score values are calculated as follows

$$P = \frac{TP}{TP+FP}, \; R = \frac{TP}{TP+FN}, \; F1\text{-}score = \frac{2\cdot(P\cdot R)}{(P+R)}.$$

Lastly, SSE as the Sum of Squared Errors for β has been calculated for all methods to measure of the general accuracy or goodness of fit of the penalized models where $SSE = \sum_{i=1}^{p}(\beta_i - \hat{\beta}_i)^2$ where β is the true coefficients.

500 random replications of the simulations have been performed. Every simulated data set has been split into a test set (as 30%) and a training set (70%) throughout each iteration of the simulation. λ and α have been selected from the training set, and the estimators have been computed using the training set. After models have been constructed using the training dataset, performance assessment criteria have been computed on the test sets and the median value of these criteria has been provided. All simulation data and analysis have been performed in R Version version 4.3.1.

21.3.2 Simulation Findings

The variable selection performances of AdaElastic-net, Elastic-net, LASSO, and AdaLASSO methods in logistic regression models with high dimensional data were assessed using performance assessment criteria such as recall (R), precision (P), F1-score, MSSE, TP, TN, FP, and FN. This comparison was conducted across eight different simulated datasets over 500 iterations. The median values of these criteria have been computed for each method under each simulation setting. Table 21.1 illustrates the simulation results of all the compared penalized methods for each setting.

AdaElastic-net in each correlation and dimension has returned the highest F1-score compared to all other penalized approaches. This observed performance of AdaElastic-net for the logistic model with high dimensional data aligns with the literature indication [5]. With a lower F1-score, LASSO presents a poor performance in high correlation that supports LASSO's limitations in handling highly correlated variables. In other words, in the presence of multicollinearity, the variable selection performance of LASSO was very weak. F1-score indicates the balance of recall and precision. Further, AdaLASSO has shown better performance in the presence of low and medium correlation while Elastic-net has demonstrated strong performance in the existence of multicollinearity. Moreover, as dimensionality increases, all methods have tended to exhibit a decrease in F1-score values. In the case of grouped variables, after AdaElastic-net, elastic-net yielded the best result, followed by AdaLASSO, while LASSO gave an unsuccessful performance.

On the other hand, MSSE gives an idea of the model's accuracy and a low value of MSSE indicates a good performance of the method. As collinearity degree (r) increases, The MSSE has tended to increase for LASSO, highlighting the failure of this method in this context. AdaLASSO method gave the lowest MSSE value only in setting 1 where low correlation. However, except in this setting, as dimensionality and collinearity degree increases, the MSSE value for this method has decreased. AdaElastic-net has continuously generated the lowest MSSE in all the compared penalized methods in all settings except setting 1. Moreover, the MSSE values for all approaches except LASSO have declined as the correlation rises while the dimensionality stays constant. When there is a high correlation or grouped effecting, the LASSO is not enough reliable and tends to randomly choose certain essential factors while neglecting the remaining relevant factors. Because of this, it is not advised to utilize the LASSO approach for variable selection in high-dimensional logistic regression when there is a strong correlation or grouped variable situation. A-ENet and A-LASSO provide satisfactory results when the other techniques' performances are examined based on the MSSE values. In contrast, it has been shown that the Elastic-net technique performs better at higher correlations and performs badly at lower correlations ($r = 0.30$).

The TN and FN rates allow us to assess how effectively the methods perform variable selection successfully. All methods demonstrated improved performance in terms of TN values as the correlation increases. Especially, AdaElastic-Net

Table 21.1 Simulation results for all settings through performance assessment criteria

Settings	Methods	MSSE	TN	FP	FN	TP	F1-score	Recall	Precision
Setting I									
p : 200	AdaElastic-net	5.401	159	21	0	20	0.656	1	0.488
r : 0.30	Elastic-net	6.104	108	72	2	18	0.327	0.900	0.200
	LASSO	5.625	129	51	2	18	0.405	0.900	0.261
	AdaLASSO	5.247	155	25	4	16	0.524	0.800	0.390
Setting II									
p : 200	AdaElastic-net	4.382	173	7	0	20	0.851	1	0.741
r : 0.60	Elastic-net	5.559	127	53	2	18	0.396	0.900	0.254
	LASSO	4.982	137	43	5	15	0.385	0.750	0.259
	AdaLASSO	4.611	164	16	4	16	0.615	0.800	0.500
Setting III									
p : 200	AdaElastic-net	3.705	177	3	0	20	0.930	1	0.870
r : 0.90	Elastic-net	4.383	154	26	0	20	0.606	1	0.435
	LASSO	5.411	146	34	6	14	0.412	0.700	0.292
	AdaLASSO	3.990	169	11	8	12	0.558	0.600	0.522
Setting IV									
p : 300	AdaElastic-net	7.136	253	27	0	20	0.597	1	0.426
r : 0.30	Elastic-net	8.772	170	110	3	17	0.232	0.850	0.134
	LASSO	8.380	204	76	4	16	0.286	0.800	0.174
	AdaLASSO	7.801	226	54	6	14	0.318	0.700	0.206
Setting V									
p : 300	AdaElastic-net	7.002	271	9	0	20	0.817	1	0.690
r : 0.60	Elastic-net	7.621	209	71	2	18	0.330	0.900	0.202
	LASSO	7.234	234	46	6	14	0.350	0.700	0.233
	AdaLASSO	7.188	241	39	6	14	0.383	0.700	0.264
Setting VI									
p : 300	AdaElastic-net	4.755	276	4	0	20	0.909	1	0.833
r : 0.90	Elastic-net	7.103	269	11	0	20	0.784	1	0.645
	LASSO	7.622	257	23	9	11	0.408	0.550	0.324
	AdaLASSO	5.698	272	8	11	9	0.486	0.450	0.529
Setting VII									
Grouped	AdaElastic-net	4.944	174	6	0	20	0.869	1	0.769
variables	Elastic-net	6.103	158	22	0	20	0.645	1	0.476
p : 200	LASSO	6.663	138	42	8	12	0.324	0.600	0.222
	AdaLASSO	6.030	164	16	10	10	0.435	0.500	0.385
Setting VIII									
Grouped	AdaElastic-net	6.142	270	10	0	20	0.800	1	0.667
variables	Elastic-net	7.925	265	15	0	20	0.727	1	0.571
p : 300	LASSO	9.642	233	47	10	10	0.259	0.500	0.175
	AdaLASSO	8.574	261	19	12	8	0.340	0.400	0.296

has consistently returned the highest TN values among all simulation settings. Following this, AdaLASSO performed well in terms of TN values in settings I-VI. Although in low and moderate correlation settings, LASSO has outperformed Elastic-net, in high correlation settings (when multicollinearity exists), Elastic-net has exhibited superior performance in terms of TN values, demonstrating a higher number of correctly identified zero coefficients compared to LASSO. In the setting of grouped variables, AdaElastic-net has been followed by the Elastic-net method, and subsequently, the AdaLASSO method because of strong correlation. LASSO has exhibited the lowest performance in the grouped variable situation. Subsequently, LASSO has returned the lowest TN value. When examining FN values, which represent the number of incorrectly identifying non-zero coefficients as zero, it has been observed that AdaElastic-net consistently achieves a value of 0 in each simulation setting, indicating a highly successful variable selection performance. The method following closely was Elastic-net, reaching a value of 0 in high correlation settings. However, AdaLASSO and LASSO have not provided satisfactory results in terms of FN values. Moreover, in the grouped variable situation, as dimensionality increased, FN values tended to show an increasing trend. Since TN+FP equals the number of zeros in the correct model, and FN+TP equals the number of non-zeros in correct models, the interpretation of FP and TP values in each setting for each method aligns with the aforementioned observations. Furthermore, while the TN values can be interpreted proportionally in the same way as Precision, FN values can be inversely interpreted in the same manner as Recall.

The “oracle property” in the variable selection process describes the optimal scenario in which a variable selection method may accurately pick the truly significant variables from a wider range of variables. TN values are used to assess the techniques that use the oracle property in the variable selection process. According to TN values, it has been observed that AdaElastic-net possesses this important oracle property. Additionally, it can be stated that the AdaLASSO method also has this significant feature. Moreover, the literature has repeatedly noted that the Elastic-net and LASSO methods do not have this property.

21.4 Conclusion

In this study, the variable selection performances of penalized methods, namely LASSO, Elastic-net, Adaptive Lasso, and Adaptive Elastic-net, have been compared and examined in logistic regression within different scenarios through an extensive simulation study in high-dimensional data. As a result, logistic regression with Adaptive Elastic-net has generally demonstrated superior performance in the variable selection process throughout all investigated high-dimensional simulation settings. On the other hand, Adaptive Lasso produced successful results in simulation settings with low and moderate correlations, while LASSO generally showed a moderate variable selection performance only in scenarios with low correlation. However, when there is a multicollinearity problem in the data, the LASSO method should not be preferred for variable selection in high-dimensional data. Elastic-net method can be preferred for variable selection in high dimensional data, especially in situations with grouped variables and the presence of multicollinearity, or in other words, scenarios with a high correlation between variables. As a future study, the performances of these embedded penalized methods will be applied and investigated on different types of regression models. It is also planned to apply these methods to real datasets and compare them with different methods.

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References

1. Gnana, D.A.A., Balamurugan, S.A.A., Leavline, E.J.: Literature review on feature selection methods for high-dimensional data. Int. J. Comput. Appl. **136**, 9–17 (2016)
2. Silaich, S., Gupta, S.: Feature selection in high dimensional data: a review. In: Kumar, S., Sharma, H., Balachandran, K., Kim, J.H., Bansal, J.C. (eds.) Third Congress on Intelligent Systems. CIS 2022. Lecture Notes in Networks and Systems, vol. 608. Springer, Singapore (2023)
3. Algamal, Z.Y., Lee, M.H.: A two-stage sparse logistic regression for optimal gene selection in high-dimensional microarray data classification. Adv. Data Anal. Classif. **13**, 753–771 (2019)
4. Biswas, S., Bordoloi, M., Purkayastha, B.: Review on feature selection and classification using neuro-fuzzy approaches. Int. J. Appl. Evol. Comput. **7**, 28–44 (2016)
5. Breiman, L.: Random forests. Mach. Learn. **45**, 5–32 (2001)
6. Friedman, J.H.: Greedy function approximation: A gradient boosting machine. Ann. Stat. **29**, 1189–1232 (2001)
7. Tibshirani, R.: Regression shrinkage and selection via the lasso. J. R. Stat. Soc. Ser. B Stat. Methodol. **58**, 267–288 (1996)
8. Zou, H.: The adaptive lasso and its oracle properties. J. Am. Stat. Assoc. **101**, 1418–1429 (2006)
9. Zou, H., Hastie, T.: Regularization and variable selection via the elastic net. J. R. Stat. Soc. Ser. B Stat. Methodol. **67**(2), 301–320 (2005)
10. Zou, H., Zhang, H.H.: On the adaptive elastic net with a diverging number of parameters. Ann. Stat. **37**, 1733–1751 (2009)
11. Schwarz, D.F., König, I.R., Ziegler, A.: On safari to Random Jungle: a fast implementation of random forests for high-dimensional data. Bioinformatics **26**, 1752–1758 (2010)
12. Kubus, M.: The problem of redundant variables in random forests. Acta Univ. Lodz. Folia Oecon. **6**, 7–16 (2018)
13. Blagus, R., Lusa, L.: Gradient boosting for high-dimensional prediction of rare events. Comput. Stat. Data. Anal. **113**, 19–37 (2017)
14. Sancar, N., Onakpojeruo, E.P., Inan, D., Ozsahin, D.U.: Adaptive elastic net based on modified PSO for variable selection in Cox model with high-dimensional data: a comprehensive simulation study. IEEE Access **11**, 127302–127316 (2023)
15. Fan, J., Li, R.: Variable selection via nonconcave penalized likelihood and its oracle properties. J. Am. Stat. Assoc. **96**, 1348–1360 (2001)

16. Xiao, N., Xu, Q.S.: Multi-step adaptive elastic-net: reducing false positives in high-dimensional variable selection. J. Stat. Comput. Simul. **85**, 3755–3765 (2015)
17. Algamal, Z.Y., Lee, M.H.: Penalized logistic regression with the adaptive LASSO for gene selection in high-dimensional cancer classification. Expert. Syst. Appl. **42**, 9326–9332 (2015)
18. Algamal, Z.Y., Lee, M.H.: High dimensional logistic regression model using adjusted elastic net penalty. Pak. J. Stat. Oper. Res. **11**, 667–676 (2015)
19. Sidey-Gibbons, J., Sidey-Gibbons, C.: Machine learning in medicine: a practical introduction. BMC Med. Res. Methodol. **19**, 1–18 (2019)

Chapter 22
Constrained Switching of Exponentially Stable Time-Delay Systems: Perspectives and Open Questions

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Abstract This chapter addresses some open questions about how to guarantee the exponential stability of switched time-delay systems by using constrained switching techniques. When extended to switched systems, the Lyapunov-Krasovskii, Lyapunov-Razumikhin, and Lyapunov-Halanay methods in time-delay systems have the potential to result in some unique average dwell-time bounds that are well-known in a delay-free context. We start presenting a result by using the family of Lyapunov-Krasovskii functionals, which is a direct extension of a well-known result in finite-dimensional systems. We also provide the results given by using the Lyapunov-Razumikhin methodology. After presenting the results in the literature, we address the question of whether the Lyapunov-Razumikhin method may have alternative extensions in switched time-delay systems. We conclude the chapter by addressing the questions of whether Lyapunov-Halanay methods can also be used to establish the exponential stability of switched time-delay systems and how a mix of all these methods can be used to establish the exponential stability of switched time-delay systems.

22.1 Introduction

Switched systems are a well-known and extensively researched class of hybrid systems that combine discrete, isolated switching events with continuous-time system orchestration. These kinds of systems are significant in and of themselves, and much study has been done on their stability characteristics [12]. For the stability and stabilization of switched systems, numerous conditions based on the existence of multiple or common Lyapunov functions are proposed (see [7, Chapter 2.1 and 3.1] and references therein). When a common Lyapunov function is absent, the stability of the switched system frequently depends on the switching signal; this

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situation is known as restricted switching and necessitates the use of numerous Lyapunov functions in the analysis.

The switched systems literature is experienced in the use of average dwell-time in the context of limited switching [7, 12]. Requirement for average dwell-time is the so-called compatibility condition; each Lyapunov function related to a subsystem must be able to be upper bounded by a scalar larger than unity times another function connected with a different subsystem. Stated differently, the leap among all functions must be captured by a scalar larger than unity for a family of functions to be compatible. It is vital to place additional restrictions on the activation time or average activation time of unstable subsystems when they show instability in order to guarantee stability in the resultant switching system [15, 16].

Different types of stability can be established in time-delay systems in a number of ways. Stability and stabilization of switching time-delay system is of particular importance since there are numerous methods for establishing different kinds of stability for time-delay systems. More specifically, [6], the Lyapunov-Krasovskii technique is a powerful tool for examining the robustness and stability of time-delay systems. Finding a functional that is appropriate and has the requisite dissipation properties together with the solutions of the related system is crucial to the success of this strategy. The target of research is now a functional, where the argument reflects the entire state history during some finite time period. This is the primary distinction between this method and the Lyapunov methodology.

On the other hand, the Lyapunov-Halanay and Lyapunov-Razumikhin methods use functions, which are scalar functions that offer an explanation of the stability and behavior characteristics of a time-delay system [5, 11]. The purpose of the usage of Lyapunov-Halanay functions or Lyapunov-Razumikhin functions is to represent the stability of the system by taking into account both the function's maximum value over a given period of time and its initial state. Unlike Lyapunov-Krasovskii techniques, which use the directional derivatives, these functions are defined by certain inequalities involving the gradient and the dynamics of the system. For further information on these various techniques, see [2–4, 9].

This chapter explores the switched time-delay systems, highlighting exponential stability as a key concept. Utilizing Lyapunov-Krasovskii, Lyapunov-Razumikhin, and Lyapunov-Halanay techniques specifically designed for time-delay systems, the investigation proceeds with distinct average dwell-time bounds. As the chapter moves from Lyapunov-Krasovskii functionals to Lyapunov-Razumikhin methods, it poses problems regarding possible extensions in switching time-delay systems. Additionally, it poses the question how Lyapunov-Halanay techniques or the mix of these three methodologies may work to guarantee the exponential stability of switched time-delay systems.

The notation is as the following. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given $\delta \geq 0$, $\mathcal{X}^n$ denotes the set of all continuous vector valued functions $\phi : [-\delta; 0] \to \mathbb{R}^n$ equipped with the norm $\|\phi\| := \sup_{\tau \in [-\delta, 0]} |\phi(\tau)|$. $\mathbb{Z}$ is the set of integers whereas $\mathbb{N}$ is the set of positive integers.

22.2 Preliminaries

Consider the switched systems consisted of time-delay systems

$$\begin{aligned} \dot{x}(t) =& f_{\sigma(t)}(x_t), \qquad a.e.\ t \geq 0, \\ x(\theta) =& x_0(\theta), \qquad \theta \in [-\delta, 0], \end{aligned} \tag{22.1}$$

where $x(t) \in \mathbb{R}^n$ is the instantaneous value state vector whereas $x_t \in \mathcal{X}^n$ denotes the state history defined over the time interval $[t-\delta, t]$ as

$$x_t(s) = x(t+s), \qquad \forall s \in [-\delta, 0].$$

Hence, $x_0 \in \mathcal{X}^n$ is the initial state history. The switching signal, a piecewise constant function of time, is defined as $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$ which is an element of the set Σ of all right-continuous, piecewise constant functions from $\mathbb{R}_{\geq 0}$ to $\mathcal{P}$ with a finite number of discontinuities. Here, the associated family of subsystems are given as

$$\dot{x}(t) = f_p(x_t), \qquad p \in \mathcal{P}, \tag{22.2}$$

where $\mathcal{P} \subset \mathbb{N}$ is some finite index set. The vector fields $f_p : \mathcal{X}^n \to \mathbb{R}^n$, $p \in \mathcal{P}$ are assumed to be Lipschitz on bounded sets of $\mathcal{X}^n$ and satisfies $f_p(0) = 0$ so that, given any $x_0 \in \mathcal{X}^n$, all subsystems admits a unique solution. For a compact formulation, we will write the unique absolutely continuous solution $x(\cdot) := x(\cdot, x_0)$ on its interval of existence for $x_0 \in \mathcal{X}^n$.

We define the global exponential stability (GES) property as the following.

Definition 22.1 (GES) The system (22.1) is said to be *globally exponentially stable (GES)* under certain switching signal σ if there exists $c, k > 0$ such that, for all $x_0 \in \mathcal{X}^n$,

$$\|x(t)\| \leq ce^{-kt}\|x_0\|, \qquad \forall t \geq 0, \tag{22.3}$$

holds along the corresponding solution $x(t)$.

In time-delay system, two special types of functional derivatives are utilized which we detail next. The upper-right Driver's derivative of a functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ is defined, for all $\phi \in \mathcal{X}^n$ and all $w \in \mathbb{R}^n$ as

$$D^+V(\phi, w) := \limsup_{h \to 0^+} \frac{V(\phi_{h,w}) - V(\phi)}{h} \tag{22.4}$$

where, for each $h \in [0, \delta)$, $\phi_{h,w} \in \mathcal{X}^n$ is given by

$$\phi_{h,w} = \begin{cases} \phi(s+h), & s \in [-\delta, -h), \\ \phi(0) + w(h+s), & s \in [-h, 0]. \end{cases} \tag{22.5}$$

In general, Driver's derivative is used in the direction of a vector field. In such a case and in switched time-delay system context, the increment is taken as $w = f_p(\phi)$, for any fixed $p \in \mathcal{P}$ with $\phi \in \mathcal{X}^n$. Moreover, it is well-known from [8, Theorem 2] that the Driver's derivative of the functional V computed at x_t corresponds almost everywhere to the upper-right Dini derivative of the function $w(t) = V(x_t)$ along the solutions of (22.2):

$$D^+_{(22.2),p} V(x_t) = D^+_{(22.2)} w(t), \quad \forall t \in [0, b),\ a.e.,$$

where

$$D^+_{(22.2)} w(t) := \limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h}. \tag{22.6}$$

We now present a list of functions and functionals differing in the way they dissipate along solutions and define global exponential stability (GES) functions/functionals.

Definition 22.2 Consider the family of switched time-delay systems (22.2) for all $p \in \mathcal{P}$. Given an index set $\mathcal{P}$,

- a family of Lipschitz continuous functionals $V_p : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$, $p \in \mathcal{P}$ on bounded subsets of $\mathcal{X}^n$, is said to be a family of GES Lyapunov-Krasovskii (GES LKFs) with $(k_3, \mathcal{P})$ if there exist $k_1, k_2, k_3 \geq 0$, $\mu \geq 1$ such that the following hold

$$V_p(\phi) \leq \mu V_q(\phi),\ \forall p, q \in \mathcal{P},\ \forall \phi \in \mathcal{X}^n, \tag{22.7}$$

$$k_1 |\phi(0)|^2 \leq V_p(\phi) \leq k_2 \|\phi\|^2,\ \forall p \in \mathcal{P},\ \forall \phi \in \mathcal{X}^n, \tag{22.8}$$

$$D^+_{(22.2),p} V_p(\phi) \leq -k V_p(\phi),\ \forall p \in \mathcal{P},\ \forall \phi \in \mathcal{X}^n, \tag{22.9}$$

- a family of continuously differentiable functions $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $p \in \tilde{\mathcal{P}}$ is said to be a family of
  - GES Lyapunov-Halanay functions (GES LHFs) with $(k_3, k_4, \tilde{\mathcal{P}})$ if there exist $k_1, k_2, k_3, k_4 > 0$, $\mu \geq 1$ with $k_3 > k_4$ such that the following hold

$$V_p(x) \leq \mu V_q(x),\ \forall p, q \in \mathcal{P},\ \forall x \in \mathbb{R}^n, \tag{22.10}$$

$$k_1 |x|^2 \leq V_p(x) \leq k_2 |x|^2,\ \forall p \in \mathcal{P},\ \forall x \in \mathbb{R}^n, \tag{22.11}$$

$$\begin{aligned} \nabla V_p(\phi(0)) f_p(\phi) \leq & -k_1 V_p(\phi(0)) \\ & + k_2 \sup_{s \in [-\delta, 0]} V_p(\phi(s)),\ \forall p \in \mathcal{P},\ \phi \in \mathcal{X}^n, \end{aligned} \tag{22.12}$$

– GES Lyapunov-Razumikhin functions (GES LRFs) with $(\rho, k_3, \tilde{\mathcal{P}})$ if there exist $k_1, k_2, k_3 > 0$, $\mu \geq 1$ and $\rho \in (0, 1)$ such that (22.10) and (22.11) and the following hold, for all $\phi \in \mathcal{X}^n$ and all $p \in \tilde{\mathcal{P}}$,

$$\begin{aligned} V_p(\phi(0)) \geq & \rho \sup_{s\in[-\delta,0]} V_p(\phi(s)) \\ & \implies \nabla V_p(\phi(0)) f_p(\phi) \leq -k_3 V_p(\phi(0)). \end{aligned} \tag{22.13}$$

Definition 22.2 presents dissipation inequalities considering a family of functions and functionals. A GES LKF is defined when the Driver's derivative of the functional satisfies a dissipation inequality involving the LKF itself which will be used to ensure an exponential convergence for a subsystem. On the other hand, GES LHF and LRF are defined for a family of functions. In this case, these functions must satisfy specific inequalities involving the inner product of gradient and the system dynamics. Moreover, LRFs are established in an implication form, while LHFs directly consider the maximum value of the function over a time interval.

We, now, define the average dwell-time and activation time for a switching signal.

Definition 22.3 (Average Dwell-Time) For a switching signal $\sigma \in \Sigma$, we say that σ has an *average dwell-time (average dwell-time)* τ_a, if it belongs to the following set

$$\Sigma_{\tau_a} := \left\{ \sigma \in \Sigma \ : \ \exists N_0 \in \mathbb{N} \ s.t. \ N_\sigma(s,t) \leq N_0 + \frac{t-s}{\tau_a}, \ 0 \leq s < t \right\}. \tag{22.14}$$

Here, $N_\sigma(s, t)$ is the number of discontinuities of the switching signal $\sigma \in \Sigma$ on an interval (s, t) and N_0 is called the chatter bound.

As a notation, we employ the switching times on the interval $(0, t)$ by $t_1, \ldots, t_{N_\sigma(0,t)}$ for an arbitrary time $t > 0$ where $N_\sigma(0, t)$ is as defined in Definition 22.3. Moreover, the initial time is considered as $t_0 = 0$.

22.3 Recent Results and Open Questions

We now present the average dwell-time conditions to ensure GES of the switched time-delay system (22.1). The first result makes the use of the family of GES LKFs which is a direct extension of the result in [14].

Theorem 22.1 *Consider the family of subsystems* (22.2) *and suppose that there exists a family of GES LKFs with* $(k_3, \mathcal{P})$. *Then, the switched time-delay* *system* (22.1) *is GES, for every switching signal* σ *satisfying the average dwell-time bound*

$$\tau_a > \frac{\ln \mu}{k_3 - k_4}, \tag{22.15}$$

for some $k_4 \in (0, k_3)$.

The presented theorem establishes GES of a corresponding switched time-delay system by considering the family of subsystems described by (22.2) and assuming the existence of a family of GES LKFs. Here, the key insight lies in the average dwell-time bound (22.15), where the bound involves an interplay between the so-called compatibility parameter μ, and exponential convergence parameters k_3 and k_4, where k_4 is constrained within the interval $(0, k_3)$. As a result, the theorem provides a guideline to choose an average dwell-time that ensures the desired level of GES in the presence of time delays and switching dynamics.

The following sufficient conditions are given in [13] by using a different Lyapunov-Razumikhin approach.

Theorem 22.2 *Consider the family of subsystems* (22.2)*. Given an index set* $\mathcal{P}$*, a family of continuously differentiable functions* $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*,* $p \in \mathcal{P}$*, assume that there exists* $k_1, k_2 > 0$*,* $\mu \geq 1$ *such that* (22.10) *and* (22.11) *hold. Suppose also that, there exist* $k_3 > 0$ *and* $\rho > 1$ *such that the following holds for the family of functions* $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*, for all* $\phi \in \mathcal{X}^n$ *and all* $p \in \tilde{\mathcal{P}}$*,*

$$\begin{aligned} V_p(\phi(0))e^{k_3\delta} \geq & \rho \sup_{s\in[-\delta,0]} V_p(\phi(s)) \\ \implies & \nabla V_p(\phi(0)) f_p(\phi) \leq -k_3 V_p(\phi(0)). \end{aligned} \tag{22.16}$$

Then, the switched time-delay system (22.1) *is GES, for every switching signal* σ *satisfying the average dwell-time bound*

$$\tau_a > \frac{\ln \mu + k_3\delta}{k_3}. \tag{22.17}$$

Differently from Theorem 22.1, Theorem 22.2 establishes a condition for the GES of switched time-delay systems within a particular usage of Lyapunov-Razumikhin framework. The key to this result lies in the average dwell-time bound (22.17) which establishes a requirement on the temporal separation between consecutive switches, expressed in terms of the Lyapunov-like function dynamics and the time delay parameter δ.

Even though, this theorem offers practical insights into the conditions under which such systems can be exponentially stabilized, the average dwell-time bound is conservative when the time delay gets bigger. That is the reason why, the constrained switching signal can be improved by allowing a more relaxed average dwell-time bound. The time delay term in the nominator in (22.17) actually originates from the

Lyapunov-Razumikhin condition (22.16) and, when the proof of [13, Theorem 1] is followed, this directs us to ask the following question.

Question 22.1 Is it possible to replace (22.16) with (22.13) to establish GES for the switched time-delay system (22.1)? In other words, what are the alternative sufficient conditions to ensure GES for (22.1) by using GES LRF functions? Particularly, what will be the average dwell-time bound and will it be more conservative or relaxed bound than (22.17)?

As introduced before, there is an alternative approach namely Lyapunov-Halanay method, which is analyzed by GES LHFs and guarantees GES of time-delay systems. An another natural question is therefore the following:

Question 22.2 Is it possible to use GES LHFs to establish GES of the switched time-delay system (22.1)? What will be the sufficient conditions and the average dwell-time bound?

Looking at the big picture, Definition 22.2 outlines establishing GES by three major methods in time-delay systems: GES LKFs, GES LHFs and GES LRFs. In some practical applications in switched time-delay systems, there might be a need to characterize some subsystems by functionals, i.e. GES LKFs, and the rest by functions, i.e. GES LHFs and/or GES LRFs. This inquiry addresses a critical gap in our understanding of the exponential stability dynamics in switched time-delay systems, where each subsystem may inherently possess different stability characteristics which allows us to pose the last question:

Question 22.3 Is it possible to establish GES of the switched time-delay system (22.1) when the families of GES LKFs, GES LHFs and GES LRFs corresponds to different subsystems? In other words, is it possible to guarantee GES when the subsystems enjoy different kind of GES functions and functionals? What will be the sufficient conditions and the average dwell-time bound for this situation?

Answering these questions raised above may also entail a rigorous exploration of the compatibility and interaction between different stability methodologies in switched time-delay systems. The findings may contribute not only to the theoretical understanding of such systems but also offer valuable guidance for the design and analysis of complex, real-world systems characterized by diverse stability characteristics across subsystems. Furthermore, once these questions are answered, the prospective studies may guarantee more general types of stability such as global asymptotical stability of the systems without inputs or input-to-state stability related properties of the systems with inputs.

22.4 Conclusions and Perspectives

In conclusion, this chapter summarized the possible and prospective research for establishing global exponential stability (GES) in switched time-delay systems.

By examining the interplay between different families of GES LKFs, GES LHFs and GES LRFs associated with distinct subsystems, we have posed the question of whether GES can be assured when subsystems exhibit diverse stability methodologies. The theorems presented offer valuable insights, providing average dwell-time conditions under which the overall switched system remains exponentially stable, even when subsystems employ GES LKFs and an exponentially weighted type of GES LRFs.

Looking ahead, several promising future research emerge from this work. Firstly, an extension of these stability results to more complex systems with nonlinearities, uncertainties, or disturbances could enhance the applicability of the proposed methodologies. Additionally, exploring the feasibility of combining different types of Lyapunov functionals/functions within a single subsystem or across subsystems may achieve stability in diverse settings. Furthermore, investigating the implications of these stability conditions on the design of control strategies for practical applications will also be an exciting direction. Working on the alternative dissipation inequality involving the instantaneous value of the solution's norm (pointwise dissipation) and the recently proposed $\mathcal{KL}$-dissipation inequality, which combines pointwise dissipation and historywise dissipation (the dissipation inequality involving the supremum norm of the state history), could also be an interesting direction. For input-to-state exponential stability or input-to-state stability, one way to approach this inequality would be to use [1, Proposition 2] or the "implication form" dissipation of [10, Theorem 3.1].

References

1. Chaillet, A., Goksu, G., Pepe, P.: Lyapunov–Krasovskii characterizations of integral input-to-state stability of delay systems with nonstrict dissipation rates. IEEE Trans. Autom. Control **67**(7), 3259–3272 (2022)
2. Chaillet, A., Karafyllis, I., Pepe, P., Wang, Y.: The ISS framework for time-delay systems: a survey. Math. Control Signals Syst. **35**(2), 237–306 (2023)
3. Fridman, E.: Introduction to Time-delay Systems: Analysis and Control. Springer (2014)
4. Gu, K., Chen, J., Kharitonov, V.L.: Stability of Time-delay Systems. Springer Science & Business Media (2003)
5. Halanay, A.: Differential Equations: Stability, Oscillations, Time Lags, vol. 23. Academic Press (1966)
6. Krasovskii, N.N.: Stability of Motion. Stanford University Press (1963)
7. Liberzon, D.: Switching in Systems and Control. Springer Science & Business Media (2003)
8. Pepe, P.: On Liapunov–Krasovskii functionals under Caratheodory conditions. Automatica **43**(4), 701–706 (2007)
9. Pepe, P.: A nonlinear version of Halanay's inequality for the uniform convergence to the origin. Math. Control Relat. Fields **12**(3), 789–811 (2022)
10. Pepe, P., Jiang, Z.P.: A Lyapunov–Krasovskii methodology for ISS and iISS of time-delay systems. Syst. Control Lett. **55**(12), 1006–1014 (2006)
11. Razumikhin, B.S.: On the stability of systems with a delay. Prikl. Mat. Mekh. **20**, 500–512 (1956)

12. Sun, Z., Ge, S.S.: Stability Theory of Switched Dynamical Systems. Springer Science & Business Media (2011)
13. Sun, X.M., Dimirovski, G.M., Zhao, J., Wang, W.: Exponential stability for switched delay systems based on average dwell time technique and Lyapunov function method. In: 2006 American Control Conference, p. 5. IEEE (2006)
14. Vu, L., Chatterjee, D., Liberzon, D.: Input-to-state stability of switched systems and switching adaptive control. Automatica **43**(4), 639–646 (2007)
15. Yang, H., Jiang, B., Cocquempot, V.: A survey of results and perspectives on stabilization of switched nonlinear systems with unstable modes. Nonlinear Anal. Hybrid Syst. **13**, 45–60 (2014)
16. Zhai, G., Hu, B., Yasuda, K., Michel, A.N.: Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach. Int. J. Syst. Sci. **32**(8), 1055–1061 (2001)

Chapter 23
A Regularization Method for an Inverse Problem Represented by a First-Kind Integral Equation

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Abstract The solution to the direct problem of geothermic under sedimentation conditions for geothermal reservoirs is considered. The main factors forming the thermal field of sedimentary basins are taken into account in the most complete way—the consumption of heat flow energy on the base for heating of cold sedimentary material, partial shielding of heat flow due to the difference of thermophysical sediments and base rocks, heat generation in accumulating sediment, and different rates of sedimentation. The problem of calculating the value of heat flux from the foundation based on temperature observations in wells—the inverse problem of geothermic in sedimentation conditions—has also been stated.

23.1 Introduction

Problems of geothermic can be described by mathematical models, specifically a set of partial differential equations along with initial and/or boundary conditions defined in a particular domain. Models in computational geothermic quantitatively predict the outcomes when the crust and mantle deform slowly over geological time. These models often incorporate complications such as simultaneous heat transfer (e.g., thermal convection in the mantle), phase changes in the Earth's deep interior, complex rheology (viscosity, plasticity, non-Newtonian fluids), melting and migration of melts, chemical reactions (e.g., thermochemical convection), motion of solid, lateral forces, etc.

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A mathematical model relates the causal characteristics of a geothermal process to its consequences. The causal characteristics of the simulated process include, for example, the parameters of the initial and boundary conditions, the coefficients, the right side of the differential equations, as well as geometric parameters, and domains. The purpose of the direct problem is to determine the relationship between the causes and effects of the geothermal process, and therefore, to formulate a mathematical problem for a given set of parameters and coefficients.

The inverse problem of geothermic is the opposite of the direct problem. The inverse problem is stated when there is no information about the causal characteristics, but there is information about the effects of the geophysical (more specifically, geothermal) process. Inverse problems can be classified as follows: inverse time problems (e.g., to reconstruct the development of a geodynamic process); coefficient problems (e.g., determination of coefficients, right-hand sides of model equations), geothermic problems (e.g., determination of the location of heat sources in a domain or geometry of boundaries), and many others.

Inverse problems often turn out to be poorly formulated or incorrect in J. Hadamard's terminology [1]. A mathematical model for a geophysical problem should be well-established in the sense that it should have the properties of (1) existence, (2) uniqueness, and (3) stability of the solution of the problem. Problems for which at least one of these properties is not performed are called ill-posed problems. If, for example, a problem does not have property (3), then its solution is almost impossible to compute because the calculations are contaminated by inevitable errors. If the solution of a problem is not continuously dependent on the initial data, then, generally speaking, the computed solution may have nothing to do with the true solution. In the works of A.N. Tikhonov and his followers, methods for solving ill-posed problems are proposed. The essence of A.N. Tikhonov's method is the construction of regularizing families of problems, the solution of which in the limit gives the solution of the initial ill-posed problem [2]. The application of A.N. Tikhonov's method to a wide class of geodynamic problems is described in [2].

23.2 Formulation of the Three-Dimensional Inverse Problem of Geothermia

Let $D = \{(x, y, t) : x \in [0, a], y \in [0, b] + t \in [0, t^*]\}$.

The boundaries of domain D consist of the following five components

$$\Gamma_0 = \{(x, y, 0) : x \in [0, a]), y \in [0, b]\},$$

$$\Gamma_1 = \{(x, 0, t) : x \in [0, a], t \in [0, t^*)\},$$

$$\Gamma_2 = \{(0, y, t) : y \in [0, b], t \in [0, t^*)\},$$

$$\Gamma_3 = \{(x, b, t) : x \in [0, a], t \in [0, t^*)\},$$

$$\Gamma_4 = \{(a, y, t) : y \in [0, b], t \in [0, t^*)\},$$

where the initial and boundary values are known.

In the domain D we consider the heat conduction equation

$$\frac{\partial u(x, y, t)}{\partial t} = d_1 \frac{\partial^2 u(x, y, t)}{\partial x^2} + d_2 \frac{\partial^2 u(x, y, t)}{\partial y^2}, (x, y, t) \in D, \tag{23.1}$$

where d_1 *and* d_2-diffusion coefficients, u-represents temperature, t-time, $u(x, y)$-spatial coordinates, respectively.

On the boundary of Γ_1 the initial condition is set as follows

$$u(x, y, 0) = \varphi(x, y), x \in [0, a], y \in 0, b. \tag{23.2}$$

On the boundary of Γ_2 the boundary condition is set as follows

$$u(0, y, t) = \psi_1(y, t), y \in [0, b], t \in [0, t^*), \tag{23.3}$$

and

$$u(x, 0, t) = \psi_2(x, t), x \in [0, a], t \in [0, t^*), \tag{23.4}$$

respectively.

In the inverse problem discussed below the temperature distribution $u(x, y, t)$ in the domain D is determined by both: the temperatures $\theta_1(y, t)$, $\theta_2(x, t)$ and heat fluxes $q_1(y, t), q_2(x, t)$ at the boundaries Γ_3 and Γ_4, for which the Dirichlet and Neumann boundary conditions are satisfied:

$$u(x, b, t) = \theta_1(x, t), x \in [0, a], t \in [0, t^*), \tag{23.5}$$

$$u(a, y, t) = \theta_2(y, t), y \in [0, b], t \in [0, t^*), \tag{23.6}$$

$$-k_1 \frac{\partial u(x, b, t)}{\partial y} = q_1(x, t), x \in [0, a] t \in [0, t^*), \tag{23.7}$$

$$-k_2 \frac{\partial u(a, y, t)}{\partial x} = q_2(y, t), y \in [0, b] t \in [0, t^*). \tag{23.8}$$

The initial description of the inverse problem is supplemented (1)–(8) with temperature values at some fixed points $x = x_{p_1}, y = y_{p_2}$ where $p_1 \in (0, a)$, $p_2 \in (0, b)$:

$$u(x_{p_1}, y, t) = \psi_{x_{p_1}}(y, t), y \in [0, b], t \in [0, t^*), \tag{23.9}$$

$$u(x, y_{p_2}, t) = \psi_{y_{p_2}}(x, t), x \in [0, a], t \in [0, t^*). \tag{23.10}$$

If the Green's function $G(x, y, t)$ of the following problem is known

$$\frac{\partial G(x, y, t)}{\partial t} = d_1 \frac{\partial^2 G(x, y, t)}{\partial x^2} + d_2 \frac{\partial^2 G(x, y, t)}{\partial y^2}, (x, y, t) \in D,$$

$$G(x, y, 0) = 0, G(0, 0, t) = 1,$$

$$\frac{\partial G(x, b, t)}{\partial x} = \frac{\partial G(a, y, t)}{\partial y} = 0,$$

then, in accordance with Duhamel's principle, the solution of the problem (1–10) is represented as

$$u(x, y, t) = \int_0^t \frac{\partial G(x, y, t - s)}{\partial t} u(x, y, s) ds, t \in [0, t^*). \tag{23.11}$$

23.3 Method of Regularization for Integral Equations of the First Kind

The three-dimensional geothermal problem formulated in (2) admits a different formulation using linear integral equations of the first kinds

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x - \xi, y - \eta) u(\xi, \eta) d\xi d\eta = f(x, y), \tag{23.12}$$

where $-\infty < x < \infty, -\infty < y < \infty, k(x-\xi, y-\eta) = k(\xi - x, \eta - y) = K(s, t)$-is the symmetric kernel of the equation, $f(x, y)$ is the given function, $u(\xi, \eta)$ is the fast function.

When solving practical problems, integration in Eq. 23.12 is carried out only in finite limits. Therefore, we consider the following equation:

$$Au \equiv \int_{-a}^{a}\int_{-b}^{b} K(x-\xi, y-\eta)u(\xi,\eta)d\xi d\eta = f(x,y), \tag{23.13}$$

where $-b \le x \le b, -a \le y \le a,\ A : H \to H$ is a linear integral operator, H is a real Hilbert space. Naturally, the error of the transition from (7) to (8) must be admissible. Let us assume that the numbers δ_1 and δ_2 characterize the accuracy of the initial data f and the operator A in some chosen metrics. Moreover, for finite limits of integration, we will assume that the function f is known in the rectangle $[-b,\ b] \times [-a,a]$, and the function u outside this region is identically zero, i.e., finite. Then the kernel $K(s,t) = K(x-\xi, y-\eta)$ is defined in the rectangle $[-2b, 2b] \times [-2a, 2a]$, but admits an extension to the plane $R \times R$.

Problem (23.13) is an ill-posed problem [3]. Now, let us explore a regularizing algorithm of its solution based on the method of M.M.Lavrentiev [4] and the fast Fourier transform [5].

Let us consider the case when the operator A is a Hilbert-Schmidt operator, meaning the kernel of Eq. (23.13) satisfies the condition

$$Au \equiv \int_{-a}^{a}\int_{-b}^{b}\int_{-a}^{a}\int_{-b}^{b} K^2(x-\xi, y-\eta)u(\xi,\eta)d\xi d\eta dx dy < \infty, \tag{23.14}$$

and the functions $u(\xi,\eta)$ and $f(x,y)$ belong to the two-dimensional Hilbert space $L_2[-a,a;\ -b,b]$. These conditions, subject to the observance of very non-rigorous for practice constraints [2], are fulfilled for many geothermic problems reduced to Eq. (23.13).

Now, let us establish some properties of the operator A. The following statement holds.

Theorem 23.1 *If the function $K(x-\xi,\ y-\eta)$ satisfies the condition (9), then A—compact linear operator in the space $L_2[-a,a;\ -b,\ b]$ and for its norm the following estimates are true*

$$\|A\| \le \Bigg(4ab\int_{-2a}^{2a}\int_{-2b}^{2b} K(s,t)dsdt + 2a\int_{-2a}^{2a}\int_{-2b}^{2b} sK^2(s,t)dsdt$$

$$-\ 2b\int_{-2a}^{0}\int_{-2b}^{2b} tK^2(s,t)dsdt - 2a\int_{-2a}^{2a}\int_{0}^{2b} sK^2(s,t)dsdt$$

$$-2b\int\limits_{0}^{2a}\int\limits_{-2b}^{2b} tK^2(s,t)dsdt+\int\limits_{-2a}^{0}\int\limits_{-2b}^{0} stK^2(s,t)dsdt+\int\limits_{0}^{2a}\int\limits_{0}^{2b} stK^2(s,t)dsdt$$

$$-\int\limits_{-2a}^{0}\int\limits_{0}^{2b} stK^2(s,t)dsdt-\int\limits_{0}^{2a}\int\limits_{-2b}^{0} stK^2(s,t)dsdt\Bigg)^{1/2}, \tag{23.15}$$

$$\|A\| < 2\Bigg(\int\limits_{-2a}^{2a}\int\limits_{-2b}^{2b} K^2(s,t)dsdt\Bigg)^{1/2}. \tag{23.16}$$

Theorem 23.1 is a generalization of a classical result from [6] for the case of two-dimensional space L_2.

The norms (23.15) and (23.16) generalize to the two-dimensional case of the norm from [6].

Let us find the spectrum of the kernel $K(s,t)$, by performing a twofold Fourier transform of the form

$$k(\omega_1,\omega_2)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} K(s,t)exp[-i(\omega_1 s+\omega_2 t)]dsdt. \tag{23.17}$$

The corresponding inverse Fourier transform has the form:

$$K(s,t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} k(\omega_1,\omega_2)exp[-i(\omega_1 s+\omega_2 t)]d\omega_1 d\omega_2. \tag{23.18}$$

It is obvious that if $K(s,t)\in L_2$, then according to Plancherel's theorem [6] the spectrum $k(\omega_1,\omega_2)\in L_2$.

Theorem 23.2 *For the integral operator A of convolution type with a symmetric kernel $K(s,t)$ to be positive in the Hilbert space $L_2[-a,a;-b,b]$ the integral operator A of convolution type with a symmetric kernel $K(s,t)$ is positive, it is sufficient that the kernel admits an extension from the region $[-2b,\ 2b]\times[-2a,2a]$ to the whole plane $R\times R$ and the spectrum of the kernel satisfies the condition $0\le k(\omega_1,\omega_2)<\infty$.*

Let's give the scheme of the proof of the theorem. The condition of positivity of the bounded self-adjoint operator A means that

$$(Au,u)\ge 0 \text{ for any } u\in L_2[-a,a;\ -b,\ b]. \tag{23.19}$$

The boundedness of the operator A follows from estimates (9) and (10). In a real Hilbert space H, the operator A is self-adjoint due to the symmetry of the kernel $K(s,t)$. Let us write in expanded form the scalar product (23.19)

$$(Au, u) = \int_{-a}^{a}\int_{-b}^{b}\int_{-a}^{a}\int_{-b}^{b} K(x-\xi, y-\eta)u(\xi,\eta)u(x,y)d\xi d\eta dx dy \tag{23.20}$$

for any $u \in L_2[-a, a; -b, b]$.

In the expression (23.20) we substitute the value $K(s,t) = K(x-\xi, y-\eta)$ from the formula (23.18) and reverse the order of integration. Taking into account that by the condition of Theorem 23.2 the spectrum of the kernel satisfies the inequality $0 \le k(\omega_1, \omega_2) < \infty$, we obtain

$$K(Au, u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(\omega_1,\omega_2)|\varphi(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2 \ge 0, \tag{23.21}$$

where

$$\varphi(\omega_1,\omega_2) = \int_{-a}^{a}\int_{-b}^{b} u(x,y)exp[-i(\omega_1 x + \omega_2 y)]dxdy. \tag{23.22}$$

Theorem 23.2 is proved. It generalizes to two-dimensional space the corresponding statement from [6].

Note that for positiveness of the operator A at $a = \infty$ and $b = \infty$, the following conditions suffice

$$K(s,t) \in L_2[-\infty, \infty; -\infty, \infty] \text{ and } 0 \le k(\omega_1,\omega_2) < \infty.$$

It follows from [5] that in the case of positiveness of the operator A, the regularizing solution of Eq. (23.13) is a solution of the following equation

$$\int_{-a}^{a}\int_{-b}^{b} K(x-\xi, y-\eta)u(\xi,\eta)d\xi d\eta + \alpha u(x,y) = f(x,y), \tag{23.23}$$

where $\alpha = \alpha(\delta_1, \delta_2) > 0$ is the regularization parameter chosen by the nonconvexity method [5].

The solution of Eq. (23.23) is obtained using the Fourier transform. Applying it to both parts of the expression (23.23) and using the convolution theorem [5], the function will have the following form

$$\hat{u}(\omega_1, \omega_2, a, b) = \frac{\hat{f}(\omega_1, \omega_2, a, b)}{2\pi(\omega_1, \omega_2) + \alpha}, \tag{23.24}$$

where $\hat{u}(\omega_1, \omega_2, a, b)$ and $\hat{f}(\omega_1, \omega_2, a, b)$ are Fourier transforms of the functions $u(x, y)$ and $f(x, y)$, respectively, on the region $[-b, b] \times [-a, a]$.

The inverse transformation with respect to (23.24) gives an approximation to the desired solution

$$u(x, y, a, b) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{f(\omega_1, \omega_2, a, b)}{2\pi k(\omega_1, \omega_2) + \alpha} exp[i(\omega_1 x + \omega_2 y)] d\omega_1 d\omega_2, \tag{23.25}$$

for which at $\delta_1 \to 0$, $\delta_2 \to 0$, and increasing limits of integration is true

$$\lim_{\substack{\alpha(\delta_1, \delta_2) \to 0 \\ a \to \infty \\ b \to \infty}} u(x, y, \alpha, \beta) = u(x, y), \tag{23.26}$$

where $u(x, y)$ is the exact value of the quantity being sought.

For practical realization of calculations by the formula (23.25), i.e., determination of approximation to the solution, it is most rational to use computational Fourier transform (FFT) schemes [7].

Let us write (23.25) in the form of a two-dimensional inverse discrete Fourier transform [7]:

$$\hat{f}(j_1 \triangle x, j_2 \triangle y)$$

$$= \frac{1}{2\pi} \sum_{j_1=0}^{N^2-1} \sum_{j_2=0}^{N_2-1} \frac{\hat{f}(p_1 \triangle \omega_1, p_2 \triangle \omega_2) exp[p_1 \triangle \omega_1 j_1 \triangle x + p_2 \triangle \omega_2 j_2 \triangle y] x}{2\pi k(p_1 \triangle \omega_1, p_2 \triangle \omega_2) + \alpha} \triangle \omega_1 \triangle \omega_2 \tag{23.27}$$

where the expression for $k(p_1\triangle\omega_1, p_2\triangle\omega_2)$ is obtained by computing the integral (23.17) and the two-dimensional discrete Fourier transform

$$\hat{f}(p_1\triangle\omega_1, p_2\triangle\omega_2)$$

$$= \frac{1}{2\pi}\sum_{j_2=0}^{N^2-1}\sum_{j_1=0}^{N_2-1} u(j_1\triangle x, j_\triangle y) exp[-i(p_1\triangle\omega_1 j_1\triangle x + p_2\triangle\omega_2 j_2\triangle y)]\triangle x, \triangle y,$$

$$p_1 = 0.1, \ldots, N_1 - 1, p_2 = 0.1, \ldots, N_2 - 1. \tag{23.28}$$

Assuming that $\triangle x = \triangle y = 1$, then $\triangle\omega_1 = 2\pi/N_1$ and $\triangle\omega_2 = 2\pi/N_2$. Substituting these expressions into (23.27) and (23.28), we obtain the final working formulas realized by successive application of one-dimensional FFT algorithms. When solving problems for large arrays, the use of FFT allows reducing the amount of computation by two orders of magnitude compared to direct computation. For example. If $N_1 = 2^{m_1}$ and $N_2 = 2^{m_2}$, where m_1 and m_2 are some natural numbers, then to perform computations (23.28) using FFT requires approximately $N_1 N_2(m_1 + m_2)$ complex multiplications and additions instead of $N_1 N_2(N_1 + N_2)$ of the same operations in direct computations. Similar questions were analyzed in [8–11].

23.4 Conclusion

A further direction of research will be the development of methods for reconstructing land surface temperature from temperature profile measurements in boreholes for glaciers and rocks with constant environmental properties. Surface temperature reconstruction will be proposed in the form of a piecewise constant function and in the form of a segment of trigonometric Fourier series.

References

1. Hadamard, J.: Sur les problems aux derivez partieless et leur signification physique. Bull. Univ., Princeton, vol. 13 (1902)
2. Tikhonov, A.N., Arsenin, V.Y.: Methods of Solving Incorrect Problems. Nauka, Moscow (1986)
3. Ismail-Zade, A., Korotkii A., Tsepelev, I.: Data-driven Numerical Modeling in Geodynamics: Methods and Applications. Springer Nature, Heidelberg (2016)
4. Starostenko, V.N., Sastri, R.G.S.: Regularizing solution of three-dimensional problems of geophysics represented by integral equations of the first kind of convolution. Dokl. USSR Acad. Sci. **246**(5), 1074–1079 (1979)
5. Lavrentiev, M.M., Klibanov, M.V.: On one integral equation of the first kind and the inverse problem for the parabolic equation. Dokl. USSR Acad. Sci. **221**(4), 782–783 (1975)

6. Rabiner, L., Gould, B.: Theory and Application of Digital Signal Processing. Mir, Moscow (1978)
7. Kolmogorov, A.N., Fomin, S.V.: Elements of the Theory of Functions and Functional Analysis. Fizmatgiz, Moscow (1972)
8. Morozov, V.A., Kirsanova, N.N., Sysov, A.F.: In Collected Works: Numerical Analysis in Fortran. M., vol. 15 (1976)
9. Ilolov, M., Rakhmatov, J.Sh.: About initial boundary value problem for fuzzy heat conduction equation. Bull. L.N. Gumilev Eurasian Natl. Univ. Ser. Math. Inf. Mech. **2**(123), 71–75 (2018)
10. Rakhmatov, J.Sh.: Fuzzy integro-differential equation of Uryson. Rep. Natl. Acad. Sci. Tajikistan **64**, 9–10, 491–500 (2021)
11. Ilolov, M., Rakhmatov J.S.: On some three-dimensional problems of geothermia. In: Proceedings of the International Conference "Voronezh Winter Mathematical School (January 26–30, 2024)", pp. 109–111. VSU Publishing House, Voronezh (2024)

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