XYZ Press

MATHEMATICAL REFLECTIONS

two special years (2014–2015)

Titu Andreescu
EDITOR

Preface

Inspired by appreciative and constructive feedback from our faithful readers, I present Mathematical Reflections: two special years, a compilation and revision of the 2014 and 2015 volumes from the online journal of the same name. Since its inception in January 2006, the journal has attracted readers and contributors from all over the world. It successfully brings together enthusiasts with different mathematical and cultural backgrounds for the common purpose of making mathematics even more elegant and exciting. The journal publishes six issues each year.

This book is aimed at high school students, participants in math competitions, undergraduates, and anyone who has a passion for mathematics. Many of the problems, solutions, and articles were submitted by passionate readers, and they all require creativity, experience, and comprehensive mathematical knowledge. In publishing this volume, my efforts were especially geared towards correcting and improving on many of the solutions and articles so that audience can enjoy them even more.

The articles herein focus on a variety of interesting topics outside of mainstream curriculum. Students can expand their mathematical horizons through material outside the scope of their regular classes. For instructors, the articles provide an intriguing opportunity to move away from a structured curriculum and guide students through to the invaluable moments of discovery. All of the featured problems are original. In order to make the material more accessible to the readers, this book as well as the journal is organized by the mathematical ability required to solve the problems. The junior section presents introductory problems (though they are not necessarily easy). The senior and Olympiad sections are for students preparing for national or international contests such as the United States of America Mathematical Olympiad (USAMO) or the International Mathematical Olympiad (IMO). Lastly, the undergraduate section offers college students a unique opportunity to solve non-routine problems in areas such as linear algebra, calculus, and graph theory.

This book could not have seen the light of day without the loyal readers and collaborators of the online journal. I would like to thank them all and express my gratitude for their continuous support. I sincerely hope that others will follow in their footsteps and continue to carry the baton so that the mission of the journal to offer the opportunity for mathematics enthusiasts to publish original and interesting work will be fulfilled in future years as well.

I would also like to thank Maxim Ignatiuc and Sean Elliott for their help in putting together the material submitted by our contributors. Many thanks to Gabriel Dospinescu for his pertinent observations. And a special thank you to Richard Stong for the many improvements in the manuscript.

If you are interested in reading the journal, please visit its website: `http://awesomemath.org/mathematical-reflections/`. Readers may submit articles, problems, and solutions to `reflections@awesomemath.org`.

Proceeds from the sale of this book will be used to sustain the journal in future years. Enjoy the problems and articles!

Titu Andreescu

Contents

1.1 Junior Problems

J289. Let a be a real number such that $0 \leq a < 1$. Prove that

$$\left\lfloor a\left(1+\left\lfloor\frac{1}{1-a}\right\rfloor\right)\right\rfloor+1=\left\lfloor\frac{1}{1-a}\right\rfloor.$$

Proposed by Arkady Alt, San Jose, California, USA

J290. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$\sqrt[3]{13a^3+14b^3}+\sqrt[3]{13b^3+14c^3}+\sqrt[3]{13c^3+14a^3}\geq 3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J291. Let ABC be a triangle such that $\angle BCA = 2\angle ABC$ and let P be a point in its interior such that $PA = AC$ and $PB = PC$. Evaluate the ratio of areas of triangles PAB and PAC.

Proposed by Panagiote Ligouras, Noci, Italy

J292. Find the least real number k such that for every positive real numbers x, y, z, the following inequality holds:

$$\prod_{cyc}(2xy+yz+zx)\leq k(x+y+z)^6.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

J293. Find all positive integers x, y, z such that

$$(x + y^2 + z^2)^2 - 8xyz = 1.$$

Proposed by Aaron Doman, University of California, Berkeley, USA

J294. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$1 \leq (a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \leq 7.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

J295. Let a, b, c be positive integers such that $(a-b)^2+(b-c)^2+(c-a)^2 = 6abc$. Prove that $a^3 + b^3 + c^3 + 1$ is not divisible by $a + b + c + 1$.

Proposed by Mihály Bencze, Braşov, Romania

J296. Several positive integers are written on a board. At each step, we can pick any two numbers u and v, where $u \geq v$, and replace them with $u+v$ and $u-v$. Prove that after a finite number of steps we can never obtain the initial list of numbers.

Proposed by Marius Cavachi, Constanţa, Romania

J297. Let a, b, c be digits in base $x \geq 4$ (with at most one being zero). Prove that

$$\frac{\overline{ab}}{\overline{ba}} + \frac{\overline{bc}}{\overline{cb}} + \frac{\overline{ca}}{\overline{ac}} \geq 3,$$

where all numbers are written in base x.

Proposed by Titu Zvonaru, Comăneşti and Neculai Stanciu, Buzău, Romania

J298. Consider a right angle $\angle BAC$ and circles $\omega_1, \omega_2, \omega_3, \omega_4$ passing through A. The centers of circles ω_1 and ω_2 lie on ray AB and the centers of circles ω_3 and ω_4 lie on ray AC. Prove that the four points of intersection, other than A, of the four circles are concyclic.

Proposed by Nairi Sedrakyan, Yerevan, Armenia

J299. Prove that no matter how we choose n numbers from the set $\{1, 2, \ldots, 2n\}$, one of them will be a square-free integer.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J300. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{\sqrt{2a^2+16ab+7b^2}+c}+\frac{c+a}{\sqrt{2b^2+16bc+7c^2}+a}+\frac{a+b}{\sqrt{2c^2+16ca+7a^2}+b}\geq 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J301. Let a and b be nonzero real numbers such that $ab \geq \frac{1}{a}+\frac{1}{b}+3$. Prove that

$$ab \geq \left(\frac{1}{\sqrt[3]{a}}+\frac{1}{\sqrt[3]{b}}\right)^3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J302. Given that the real numbers x, y, z satisfy $x+y+z=0$ and

$$\frac{x^4}{2x^2+yz}+\frac{y^4}{2y^2+zx}+\frac{z^4}{2z^2+xy}=1,$$

determine, with proof, all possible values of $x^4+y^4+z^4$.

Proposed by Răzvan Gelca, Texas Tech University, USA

J303. Let ABC be an equilateral triangle. Consider a diameter XY of the circle centered at C which passes through A and B such that lines AB and XY as well as lines AX and BY meet outside this circle. Let Z be the point of intersection of AX and BY. Prove that

$$AX \cdot XZ + BY \cdot YZ + 2CZ^2 = XZ \cdot YZ + 6AB^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J304. Let a, b, c be real numbers such that $a+b+c=1$. Let M_1 be the maximum value of $a+\sqrt{b}+\sqrt[3]{c}$ and let M_2 be the maximum value of $a+\sqrt{b+\sqrt[3]{c}}$. Prove that $M_1=M_2$ and find this value.

Proposed by Aaron Doman, University of California, Berkeley, USA

J305. Consider a triangle ABC with $\angle ABC = 30^\circ$. Suppose the length of the angle bisector from vertex B is twice the length of the angle bisector from vertex A. Find the measure of $\angle BAC$.

Proposed by Mircea Lascu and Marius Stanean, Zalău, Romania

J306. Let S be a nonempty set of positive real numbers such that for any a, b, c in S, the number $ab+bc+ca$ is rational. Prove that for any a and b in S, $\frac{a}{b}$ is a rational number.

Proposed by Bogdan Enescu, Buzău, Romania

J307. Prove that for each positive integer n there is a perfect square whose sum of digits is equal to 4^n.

Proposed by Mihály Bencze, Braşov, Romania

J308. Are there triples (p, q, r) of primes for which $(p^2-7)(q^2-7)(r^2-7)$ is a perfect square?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J309. Let n be an integer greater than 3 and let S be a set of n points in the plane that are not the vertices of a convex polygon and such that no three are collinear. Prove that there is a triangle with the vertices among these points having exactly one other point from S in its interior.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J310. Alice puts checkers in some cells of an 8×8 board such that:

a) there is at least one checker in any 2×1 or 1×2 rectangle;

b) there are at least two adjacent checkers in any 7×1 or 1×7 rectangle.

Find the least amount of checkers that Alice needs to satisfy both conditions.

Proposed by Roberto Bosch Cabrera, Havana, Cuba

J311. Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\frac{a(b^2+3)}{3c^2+1}+\frac{b(c^2+3)}{3a^2+1}+\frac{c(a^2+3)}{3b^2+1}\geq 3.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J312. Let ABC be a triangle with circumcircle Γ and let P be a point in its interior. Let M be the midpoint of side BC and let lines AP, BP, CP intersect BC, CA, AB at X, Y, Z, respectively. Furthermore, let line YZ intersect Γ at points U and V. Prove that M, X, U, V are concyclic.

Proposed by Cosmin Pohoata, Princeton University, USA

J313. Solve in real numbers the system of equations

$$x(y+z-x^3)=y(z+x-y^3)=z(x+y-z^3)=1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J314. Alice was dreaming. In her dream, she thought that primes of the form $3k+1$ are weird. Then she thought it would be interesting to find a sequence of consecutive integers all of which are greater than 1 and which are not divisible by weird primes. She quickly found five consecutive numbers with this property:

$$8=2^3,\quad 9=3^2,\quad 10=2\cdot 5,\quad 11=11,\quad 12=2^2\cdot 3.$$

What is the length of the longest sequence she can find?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J315. Let a, b, c be non-negative real numbers such that $a + b + c = 1$. Prove that

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} \geq \sqrt{5} + 2.$$

Proposed by Cosmin Pohoata, Columbia University, USA

J316. Solve in prime numbers the equation

$$x^3 + y^3 + z^3 + u^3 + v^3 + w^3 = 53353.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J317. In triangle ABC, the angle bisector of angle A intersects line BC at D and the circumcircle of triangle ABC at E. The external angle bisector of angle A intersects line BC at F and the circumcircle of triangle ABC at G. Prove that $DG \perp EF$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J318. Determine the functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x-y) - xf(y) \leq 1 - x$ for all real numbers x and y.

Proposed by Marcel Chiriţă, Bucharest, Romania

J319. Let $0 = a_0 < a_1 < \ldots < a_n < a_{n+1} = 1$ such that $a_1 + a_2 + \ldots + a_n = 1$. Prove that

$$\frac{a_1}{a_2 - a_0} + \frac{a_2}{a_3 - a_1} + \ldots + \frac{a_n}{a_{n+1} - a_{n-1}} \geq \frac{1}{a_n}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J320. Find all positive integers n for which $2014^n + 11^n$ is a perfect square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J321. Let x, y, z be positive real numbers such that $xyz(x+y+z) = 3$. Prove that
$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{54}{(x+y+z)^2} \geq 9.$$

Proposed by Marius Stanean, Zalău, Romania

J322. Let ABC be a triangle with centroid G. The parallel lines through a point P situated in the plane of the triangle to the medians AA', BB', CC' intersect lines BC, CA, AB at A_1, B_1, C_1, respectively. Prove that
$$A'A_1 + B'B_1 + C'C_1 \geq \frac{3}{2}PG.$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

J323. In triangle ABC,
$$\sin A + \sin B + \sin C = \frac{\sqrt{5}-1}{2}.$$
Prove that $\max(A, B, C) > 162°$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J324. Let ABC be a triangle and let X, Y, Z be the reflections of A, B, C across the opposite sides. Let X_b, X_c be the orthogonal projections of X on AC, AB, Y_c, Y_a the orthogonal projections of Y on BA, BC, and Z_a, Z_b the orthogonal projections of Z on CB, CA, respectively. Prove that X_b, X_c, Y_c, Y_a, Z_a, Z_b are concyclic.

Proposed by Cosmin Pohoata, Columbia University, USA

J325. For positive real numbers a and b, define their *perfect mean* to be half of the sum of their arithmetic and geometric means. Find how many unordered pairs of integers (a, b) from the set $\{1, 2, \ldots, 2015\}$ have their perfect mean a perfect square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J326. Let a, b, c be nonnegative real numbers. Prove that

$$\sqrt{2a^2+3b^2+4c^2}+\sqrt{3a^2+4b^2+2c^2}+\sqrt{4a^2+2b^2+3c^2} \geq (\sqrt{a}+\sqrt{b}+\sqrt{c})^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas

J327. A jeweler makes a circular necklace out of nine distinguishable gems: three sapphires, three rubies and three emeralds. No two gems of the same type can be adjacent to each other and necklaces obtained by rotation and reflection (flip) are considered to be identical. How many different necklaces can she make?

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J328. Let a, b, c be positive real numbers such that $a+b+c=2$. Prove that

$$\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab} \leq 3.$$

Proposed by An Zhen-ping, Xianyang Normal University, China

J329. Let $a_1, a_2, \ldots, a_{2015}$ be positive integers such that

$$a_1+a_2+\ldots+a_{2015}=a_1 \ldots a_{2015}.$$

Prove that among numbers $a_1, a_2, \ldots, a_{2015}$ at most nine are greater than 1.

Proposed by Titu Zvonaru, Comăneşti and Neculai Stanciu, Buzău, Romania

J330. Let $ABCD$ be a quadrilateral with centroid G, inscribed in a circle with center O, and diagonals intersecting at P. Prove that if O, G, P are collinear, then $ABCD$ is an isosceles trapezoid.

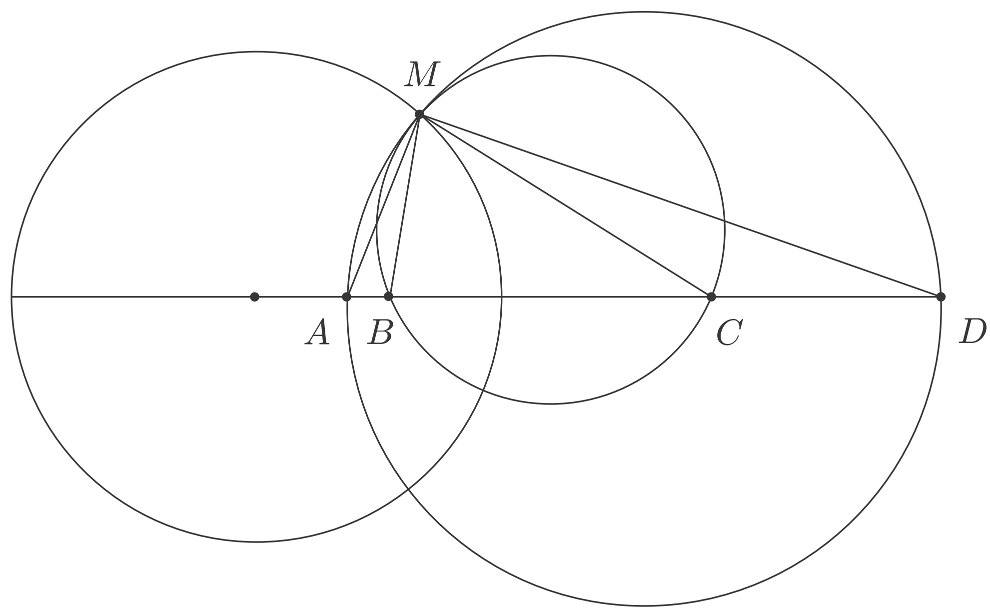
Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

3.18 Inequalities on Ratios of Radii of Tangent Circles

Some inequalities involving ratios of radii of internally tangent circles which intersect the given line in fixed points are studied. By considering special and degenerate cases of the construction, some surprising inequalities are obtained. Analogous results for externally tangent circles are also discussed.

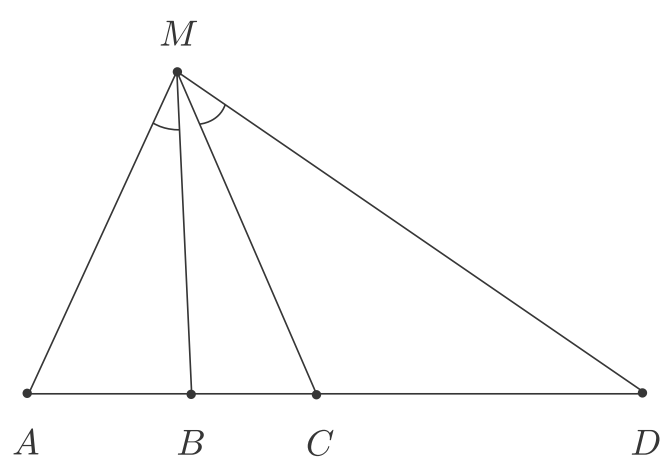
Introduction

If A, B, C, and D are fixed points on a line in the given order then the locus of points M not on line AD for which $\angle AMB = \angle CMD$ is a circle whose diameter lies on the line AD. This circle, which is sometimes called *Apollonius circle*, is also interesting for the other reason. Note that the circumscribed circles of the triangles AMD and CMB are tangent if M lies on the Apollonius circle. If we fix the circumscribed circle of AMD and move point M along this circle then the ratio $\angle CMB/\angle AMD$ decreases. So in a certain sense Apollonius circle is the locus of points M for which the ratio $\angle CMB/\angle AMD$ is maximal. On the other hand it would be interesting to find maximal and minimal values of the ratio $\angle CMB/\angle AMD$ if M is on the Apollonius circle. We have not succeeded in solving the last problem completely. For more information on the history of the question see [1]. But our investigations led to some interesting inequalities which we collected in the present paper.



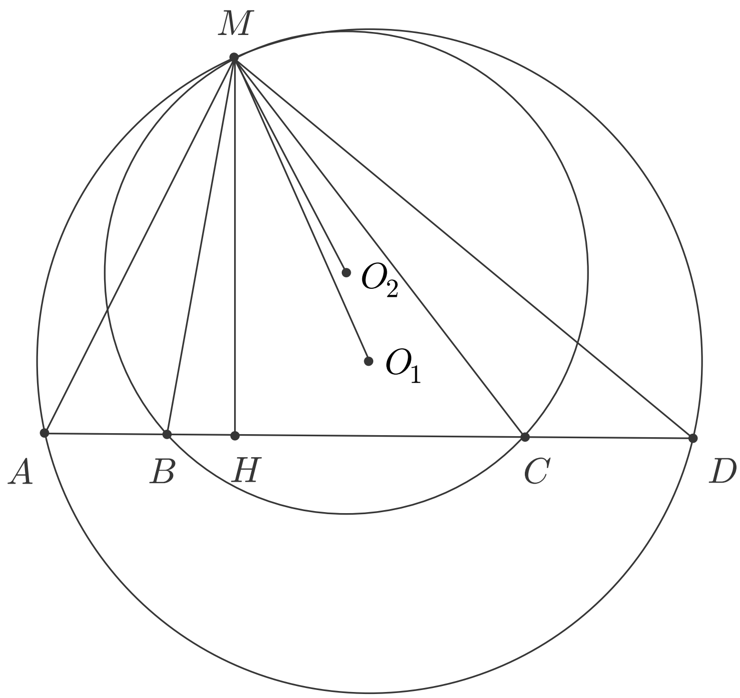
1. Theorem. *Let A, B, C, and D be points on line k in this order, and M be a point not on k such that $\angle AMB = \angle CMD$. Then*

$$\frac{\sin\angle BMC}{\sin\angle AMD} > \frac{|BC|}{|AD|}.$$



Proof. We shall first prove that circumscribed circles of triangles AMD and BMC with radii R and r respectively, are tangent at point M. Let O_1 and O_2 be circumcenters of triangles AMD and BMC. Drop perpendicular MH to line AD. It is easy to show that $\angle AMH = \angle DMO_1$ and $\angle BMH = \angle CMO_2$.

By subtracting we obtain $\angle AMB = \angle CMD + \angle O_1MO_2$. Note that $\angle AMB = \angle CMD$. It follows that points M, O_1 and O_2 are collinear.



Therefore these circles are tangent at point M. We see that $R > r$. By sine theorem

$$\frac{\sin\angle BMC}{\sin\angle AMD} = \frac{R}{r}\cdot\frac{BC}{AD} > \frac{|BC|}{|AD|}.$$

2. Theorem. *Let A, B, C, and D be points on line k in this order, and M be a point not on k such that $\angle AMB = \angle CMD$. Then*

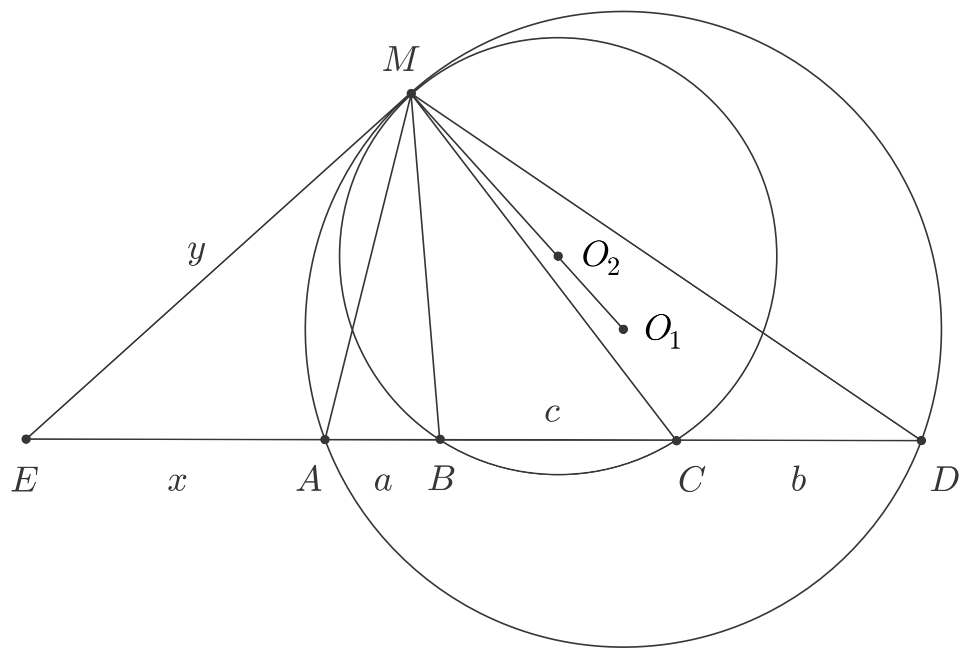
$$\frac{|BC|\cdot|AD|}{\left(\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}\right)^2} \geq \frac{\sin\angle BMC}{\sin\angle AMD}$$

$$\geq \frac{|BC|\cdot|AD|}{\left(\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}\right)^2}.$$

Proof. Suppose first that $|AB| < |CD|$. Let the common tangent line of the circumcircles of AMD and BMC at point M intersect line AD at point E.

Denote $|AB| = a$, $|CD| = b$, $|BC| = c$, $|EA| = x$ and $|EM| = y$. It follows that

$$y^2 = x(x+a+b+c) \quad \text{and} \quad y^2 = (x+a)(x+a+c).$$



By subtracting we obtain

$$x = \frac{a(a+c)}{b-a}.$$

Putting this in either of the previous equalities gives

$$y = \frac{\sqrt{ab(a+c)(b+c)}}{b-a}.$$

Denote $\angle MEA = \mu$. Drop the perpendiculars O_1K and O_2L to line AD. We obtain

$$EH = y\cos\mu,\ AK = KD = \frac{a+b+c}{2},\ BL = LC = \frac{c}{2},$$

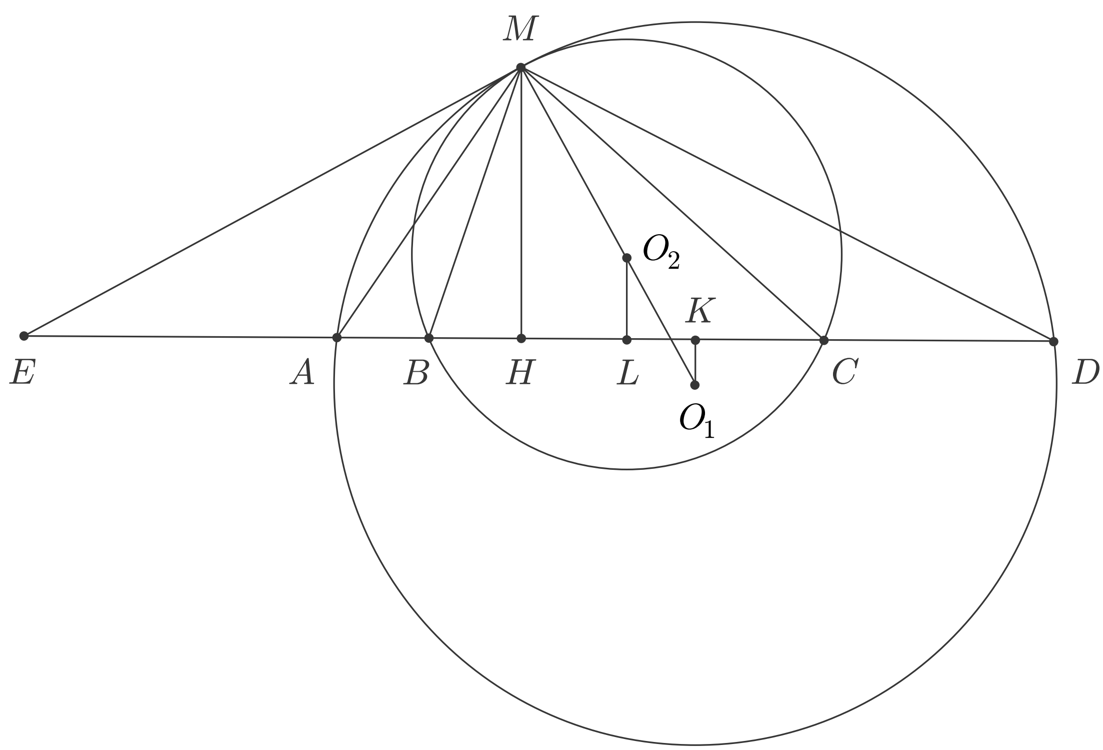
$$HL = x + a + \frac{c}{2} - y\cos\mu,\ HK = x + \frac{a+b+c}{2} - y\cos\mu.$$

Consequently,

$$\frac{R}{r} = \frac{x + \dfrac{a+b+c}{2} - y\cos\mu}{x + a + \dfrac{c}{2} - y\cos\mu}.$$

Since $a < b$, fraction R/r decreases as angle μ increases from 0 to π. Therefore

$$\frac{x + \dfrac{a+b+c}{2} + y}{x + a + \dfrac{c}{2} + y} \le \frac{R}{r} \le \frac{x + \dfrac{a+b+c}{2} - y}{x + a + \dfrac{c}{2} - y}.$$



It remains only to simplify the expressions on both sides of this double inequality.

$$\frac{x + \dfrac{a+b+c}{2} \pm y}{x + a + \dfrac{c}{2} \pm y} = \frac{\dfrac{a(a+c)}{b-a} + \dfrac{a+b+c}{2} \pm \dfrac{\sqrt{ab(a+c)(b+c)}}{b-a}}{\dfrac{a(a+c)}{b-a} + a + \dfrac{c}{2} \pm \dfrac{\sqrt{ab(a+c)(b+c)}}{b-a}}$$

$$\frac{2a(a+c)+(b-a)(a+b+c)\pm 2\sqrt{ab(a+c)(b+c)}}{2a(a+c)+(b-a)(2a+c)\pm 2\sqrt{ab(a+c)(b+c)}}$$

$$=\frac{a(a+c)+b(b+c)\pm 2\sqrt{ab(a+c)(b+c)}}{a(b+c)+b(a+c)\pm 2\sqrt{ab(a+c)(b+c)}}=\frac{\left(\sqrt{a(a+c)}\pm\sqrt{b(b+c)}\right)^2}{\left(\sqrt{a(b+c)}\pm\sqrt{b(a+c)}\right)^2}.$$

By multiplying the numerator with its conjugate and then dividing with the same conjugate we obtain

$$\frac{\left(\sqrt{a(a+c)}\pm\sqrt{b(b+c)}\right)^2}{\left(\sqrt{a(b+c)}\pm\sqrt{b(a+c)}\right)^2}$$

$$=\frac{\left(\sqrt{a(a+c)}\pm\sqrt{b(b+c)}\right)^2}{\left(\sqrt{a(b+c)}\pm\sqrt{b(a+c)}\right)^2}\cdot\frac{\left(\sqrt{a(a+c)}\mp\sqrt{b(b+c)}\right)^2}{\left(\sqrt{a(a+c)}\mp\sqrt{b(b+c)}\right)^2}$$

$$=\left(\frac{a(a+c)-b(b+c)}{a\sqrt{(a+c)(b+c)}-b\sqrt{(a+c)(b+c)}\pm(a+c)\sqrt{ab}\mp(b+c)\sqrt{ab}}\right)^2$$

$$=\left(\frac{(a+b+c)(a-b)}{\left(\sqrt{(a+c)(b+c)}\pm\sqrt{ab}\right)(a-b)}\right)^2=\left(\frac{a+b+c}{\sqrt{(a+c)(b+c)}\pm\sqrt{ab}}\right)^2.$$

Finally,

$$\frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}+\sqrt{ab}\right)^2}\le\frac{\sin\angle BMC}{\sin\angle AMD}=\frac{R}{r}\cdot\frac{c}{a+b+c}$$

$$\le\frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}-\sqrt{ab}\right)^2}.$$

The case $|AB|>|CD|$ is analogous. For the case $|AB|=|CD|$ $(b=a)$ one must pass to the limit in the last double inequality by letting b tend to a.

Note. The following problem is open: Prove that

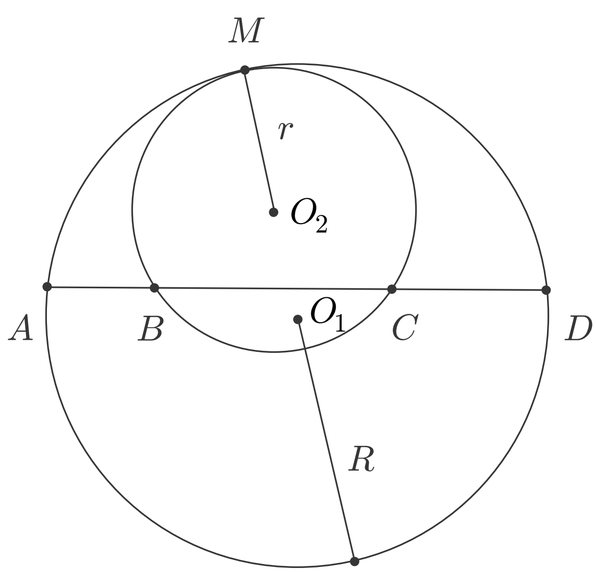
$$\frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}+\sqrt{ab}\right)^2} \le \frac{\angle BMC}{\angle AMD}.$$

By proving this inequality the following chain of inequalities will be completed [1]:

$$\frac{c}{a+b+c} \le \frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}+\sqrt{ab}\right)^2} \le \frac{\angle BMC}{\angle AMD}$$
$$\le \frac{\sin\angle BMC}{\sin\angle AMD} \le \frac{c(a+b+c)}{\left(\sqrt{(a+c)(b+c)}-\sqrt{ab}\right)^2}.$$

3. Corollary. *Let R and $r < R$ be the radii of two circles which are tangent at point M. Chord AD of greater circle intersects the other circle at points B and C. Then*

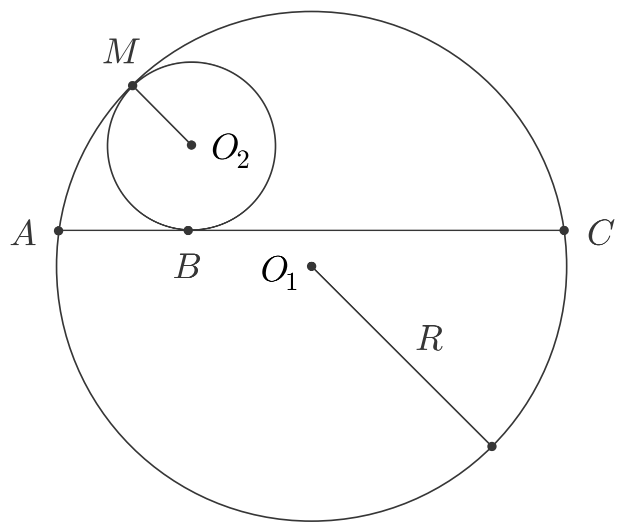
$$\left(\frac{|AD|}{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}}\right)^2 \le \frac{R}{r} \le \left(\frac{|AD|}{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}\right)^2.$$



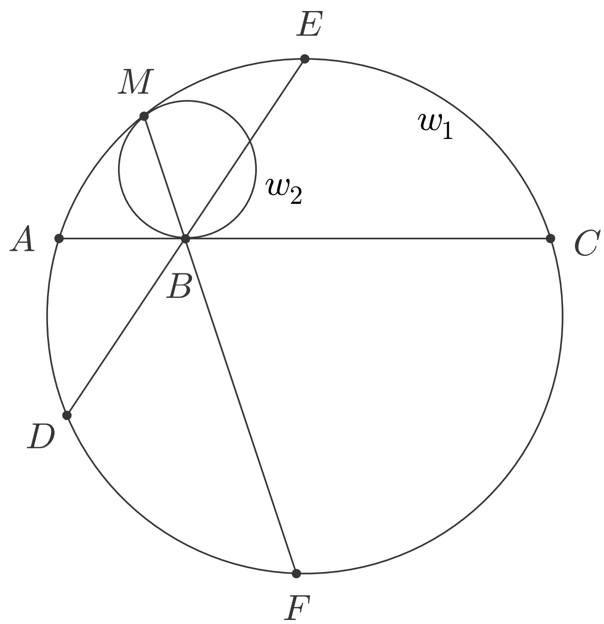
Note. In the notations of previous problems this inequality can also be written as
$$\frac{a+b+c}{\sqrt{(a+c)(b+c)}+\sqrt{ab}} \le \sqrt{\frac{R}{r}} \le \frac{a+b+c}{\sqrt{(a+c)(b+c)}-\sqrt{ab}}.$$

4. Corollary. [3] *Let R and $r < R$ be the radii of two circles which are tangent at point M. Chord AC of greater circle is tangent to the other circle at point B. Then*
$$\frac{|AC|}{2\sqrt{|AB|\cdot|BC|}} \le \sqrt{\frac{R}{r}}.$$

Note. This follows from previous problem (simply put $|BC| = 0$). It is interesting that the above inequality gives a lower bound on $\sqrt{\frac{R}{r}}$ as a ratio of the arithmetic mean $\frac{|AB|+|BC|}{2}$ and geometric mean $\sqrt{|AB|\cdot|BC|}$ of two segments AB and BC.

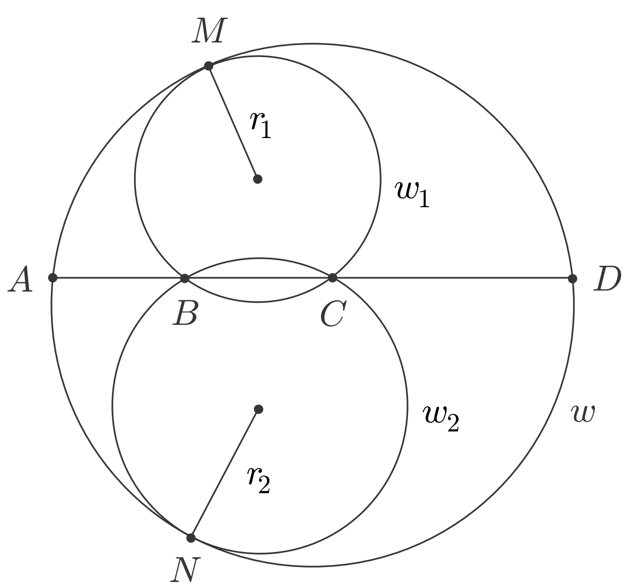


5. Corollary. *Chord AC of circle w_1 passes through midpoint B of chord DE of the same circle w_1. Circle w_2 is tangent to line AC at point B and circle w_1 at point M. Line MB intersects w_1 at point F. Then*
$$\frac{|AC|}{|DE|} \le \sqrt{\frac{|MF|}{|MB|}}.$$



6. Corollary. [2] *Two circles w_1 and w_2 intersecting at points B and C are tangent to circle w internally at points M and N, respectively. Line BC intersects circle w at points A and D. Let r_1 and r_2 be the radii of circles w_1 and w_2, respectively. Then*

$$\frac{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}} \leq \sqrt{\frac{r_1}{r_2}}.$$



Proof. Let radius of circle w be R. By the inequality in Exercise 3,

$$\left(\frac{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}{|AD|}\right)^2 \le \frac{r_1}{R},$$

$$\left(\frac{|AD|}{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}\right)^2 \le \frac{R}{r_2}.$$

Multiplying we obtain the required inequality.

Note. By replacing r_1 with r_2 and vice versa we can also obtain the upper bound for $\sqrt{\frac{r_1}{r_2}}$:

$$\sqrt{\frac{r_1}{r_2}} \le \frac{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}}{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}.$$

So in fact the following double inequality holds true

$$\frac{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}} \le \sqrt{\frac{r_1}{r_2}} \le \frac{\sqrt{|AC|\cdot|BD|}+\sqrt{|AB|\cdot|CD|}}{\sqrt{|AC|\cdot|BD|}-\sqrt{|AB|\cdot|CD|}}.$$

The following theorem and its consequence can be proved using the same method.

7. Theorem. *A circle w passing through the points A and B is externally tangent to a circle w_1. Line AB intersects the circle w_1 at points C and D. Let r_1 and R be radii of circles w_1 and w, respectively. Then*

$$\frac{|AB|}{\sqrt{|AC|\cdot|BD|}+\sqrt{|BC|\cdot|AD|}} \le \sqrt{\frac{R}{r_1}} \le \frac{|AB|}{\sqrt{|AC|\cdot|BD|}-\sqrt{|BC|\cdot|AD|}}.$$

If w_2 is another circle passing through the points C and D, and externally tangent to the circle w then

$$\frac{\sqrt{|AC|\cdot|BD|}-\sqrt{|BC|\cdot|AD|}}{\sqrt{|AC|\cdot|BD|}+\sqrt{|BC|\cdot|AD|}} \le \sqrt{\frac{r_1}{r_2}} \le \frac{\sqrt{|AC|\cdot|BD|}+\sqrt{|BC|\cdot|AD|}}{\sqrt{|AC|\cdot|BD|}-\sqrt{|BC|\cdot|AD|}},$$

where r_2 is the radius of circle w_2.
Corollary. *Let circles w and w_1 of radii R and r be externally tangent. Let the extension of chord AB of the circle w be tangent to the circle w_1 at the point C. Let CD be tangent to the circle w. Then*

$$\frac{|AB|}{2|CD|} \leq \sqrt{\frac{R}{r}}.$$

Theorem. *Let circles w and w_1 of radii R and r be externally tangent. A line through the center of circle w_1 is tangent to the circle w at the point A. Let AB be tangent to circle w_1 at the point B. Similarly, a line through the center of circle w is tangent to the circle w_1 at the point C. Let CD be tangent to circle w at the point D. Then*

$$|AB| = |CD| \geq \sqrt{Rr}.$$

Problems for further explorations

Do the last equality and the inequality hold true if the circles w and w_1 are
1) nonintersecting
2) intersecting?

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