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Advances in Functional Analysis and Operator Theory

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Contents

Preface

This volume contains the proceedings of the Special Session on Advances in Functional Analysis and Operator Theory, held July 18–22, 2022, as part of the 2nd Joint Congress of Mathematics co-organized by the American Mathematical Society, the European Mathematical Society, and the Société Mathématique de France at Université Grenoble Alpes Campus in Grenoble, France.

The papers reflect the modern interplay between differential equations, functional analysis, operator algebras, and their applications from the dynamics to quantum groups to number theory. The topics discussed are: the Sturm-Liouville and boundary value problems, axioms of quantum mechanics, C^*-algebras and symbolic dynamics, Schlesinger systems, von-Neumann algebras and low-dimensional topology, quantum permutation groups, the Jordan algebras, the Villadsen idempotents, the Kadison-Singer transforms and the Shafarevich-Tate groups of abelian varieties.

Contemporary Mathematics

Minimality conditions for Sturm-Liouville problems with a boundary condition depending affinely or quadratically on an eigenparameter

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Abstract. In the paper we study Sturm-Liouville problems with a boundary condition depending affinely or quadratically on an eigenparameter. The necessary and sufficient conditions for minimality and completeness of the chosen system of root functions of the corresponding operator were given in two forms, one with the use of special associated functions and another one with the direct use of characteristic functions. This direct method was known for the affine case and was extensively discussed in the literature. The aim of the present paper is to develop this direct method for the quadratic case and to consider the affine and quadratic cases together in a unified way.

1. Introduction

Consider the spectral problem

$$-y'' + q(x)y = \lambda y,\ 0 < x < 1, \tag{1}$$

$$y'(0)\sin\beta = y(0)\cos\beta,\ 0 \le \beta < \pi, \tag{2}$$

$$y'(1) = (a\lambda^2 + b\lambda + c)y(1), \tag{3}$$

where λ is the spectral parameter, $q(x)$ is a real valued and continuous function on the interval $[0, 1]$, and a, b, c are real. We will focus on two cases Quadratic $a \neq 0$, and Affine $a = 0, b < 0$. In the papers by Binding, Browne, Watson, and Code [**7**], [**9**] the existence and asymptotics of the eigenvalues of (1)-(3) were studied. It was proved that the eigenvalues of (1)-(3) form an infinite sequence, accumulating only at $+\infty$, and the following cases are possible:

(a) All the eigenvalues are real and simple;

(b) All the eigenvalues are simple and all, except a conjugate pair of non-real, are real;

(c) All the eigenvalues are real and all, except one double, are simple;

(d) All the eigenvalues are real and all, except one triple, are simple.

The eigenvalues λ_n $(n \ge 0)$ will be considered to be listed according to non-decreasing real part and repeated according to algebraic multiplicity. The asymptotic formulas of eigenvalues for more general polynomial case were studied in [**8**].

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For the affine decreasing case the asymptotic formula is the following

$$\lambda_n = \begin{cases} (n-1/2)^2\pi^2 + O(1) & \text{if } \beta \neq 0, \\ n^2\pi^2 + O(1) & \text{if } \beta = 0. \end{cases}$$

For the quadratic case the formula is

$$\lambda_n = \begin{cases} (n-3/2)^2\pi^2 + O(1) & \text{if } \beta \neq 0, \\ (n-1)^2\pi^2 + O(1) & \text{if } \beta = 0. \end{cases}$$

We are interested in the basis properties in $L_p(0,1)$ $(1<p<\infty)$ of the root function system of the boundary value problem (1)-(3). In order to obtain a basis consisted of eigenfunctions one needs to eliminate one or two of the eigenfunctions. This phenomenon of defect in the system of eigenfunctions is well known and was discussed in many studies [**12**], [**17**], [**21**], [**22**], [**23**]. As in [**1**] and [**2**], our main objective will be the minimality conditions for the system of root functions (cf. [**18**]). But these minimality conditions can be easily extended to basis properties in $L_p(0,1)$ $(1<p<\infty)$ (see e.g. [**4**]).

The following special problem appears in the studies of the torsional vibrations of a shaft with a disk at one end [**5**], and the vibrations of a homogeneous string with a weight at one end [**24**]:

$$-y'' = \lambda y,\ 0<x<1,$$
$$y(0)=0,\ y'(1)=b\lambda y(1),\ b>0.$$

Similar problems appear in the study of the surface viscosity of nematic liquid crystals in contact with solid surfaces [**6**], in the study of propogation of heat in a rod with concentrated heat capacity at one end [**17**]. All the eigenvalues of this special boundary value problem are real and simple. But if $b<0$ or if linear function in the boundary condition $(b\lambda+c)$ is replaced by a more general quadratic function $(a\lambda^2+b\lambda+c)$ then not allways all the eigenvalues are real and when all are real then not always all are simple. There are possibilities of a conjugate pair of non-real eigenvalues and a double or a triple eigenvalue for these more general problems. In these cases, there are many choices of the eliminated root functions (eigenfunctions or associated functions). In some cases, there are singularities when the system of root functions with some eliminated functions is not minimal or not complete. These cases can be described by some necessary and sufficient conditions. In the paper two ways of describing these singular cases will be discussed. One of them is indirect and uses specially defined associated functions. The second way is more direct and therefore more suitable for calculations. The focus of the present paper will be the description of these direct methods. The comparison of the methods was done on several examples at the end of the paper.

2. Terminology and notations

The notations and the preliminary results in this section were given in [**2**], [**4**] and some of them are included here just for completeness. We define $y(x,\lambda)$ to be the non-zero solution of (1), (2), analytic in $\lambda\in\mathbf{C}$, and we write the characteristic equation as

$$\omega(\lambda) = y'(1,\lambda) - (a\lambda^2+b\lambda+c)y(1,\lambda).$$

The eigenvalue λ_k is multiple if $\omega'(\lambda_k)=0$, in particular, we say that λ_k is a double eigenvalue if in addition $\omega''(\lambda_k)\neq 0$, and a triple eigenvalue if $\omega''(\lambda_k) = 0 \neq \omega'''(\lambda_k)$. Let y_n be an eigenfunction corresponding to eigenvalue λ_n. Note that

$y(x,\lambda) \to y(x,\lambda_n) = y_n$, uniformly in $x \in [0,1]$, as $\lambda \to \lambda_n$. If λ_k is a multiple eigenvalue ($\lambda_k = \lambda_{k+1}$) then the first order associated function y_{k+1} is defined by

$$-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k,$$

$$y'_{k+1}(0)\sin\beta = y_{k+1}(0)\cos\beta,$$

$$y'_{k+1}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+1}(1) + (2a\lambda_k + b)y_k(1).$$

Here and in the following the defintion of the associated function from [**19**] was used. Since $\omega(\lambda_k) = \omega'(\lambda_k) = 0$ then $y(x,\lambda) \to y_k$, $y_\lambda(x,\lambda) \to \widetilde{y}_{k+1}$, uniformly according to $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\widetilde{y}_{k+1}$ is one of the first order associated functions, $\widetilde{y}_{k+1} = y_{k+1} + \widetilde{c}y_k$, and $\widetilde{c} = (\widetilde{y}_{k+1}(1) - y_{k+1}(1))/y_k(1)$.

If λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$) then together with the first order associated function y_{k+1} there exists the second order associated function y_{k+2}, for which the following relations hold:

$$-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1},$$

$$y'_{k+2}(0)\sin\beta = y_{k+2}(0)\cos\beta,$$

$$y'_{k+2}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+2}(1) + (2a\lambda_k + b)y_{k+1}(1) + ay_k(1).$$

If λ_k is a triple eigenvalue $\omega''(\lambda_k) = 0$ then $y_{\lambda\lambda} \to 2\widetilde{y}_{k+2}$, uniformly in $x \in [0,1]$, as $\lambda \to \lambda_k$, where $\widetilde{y}_{k+2}$ is one of the second order associated functions corresponding to the first order associated function $\widetilde{y}_{k+1}$, $\widetilde{y}_{k+2} = y_{k+2} + \widetilde{c}y_{k+1} + \widetilde{d}y_k$, and $\widetilde{d} = (\widetilde{y}_{k+2}(1) - y_{k+2}(1) - \widetilde{c}y_{k+1}(1))/y_k(1)$.

Definition 1. *If λ_k is a double eigenvalue then let $y^*_{k+1} = y_{k+1} + c_1 y_k$, where*

$$c_1 = -\frac{\omega'''(\lambda_k)}{3\omega''(\lambda_k)} - \frac{\widehat{y}_{k+1}(1)}{y_k(1)},$$

and $\widehat{y}_{k+1} = y_{k+1} - \widetilde{c}y_k$.

Definition 2. *If λ_k is a triple eigenvalue then let $y^{\#}_{k+1} = y_{k+1} + c_2 y_k$ and $y^{\#}_{k+2} = y_{k+2} + c_2 y_{k+1} + d_1 y_k$, where*

$$c_2 = -\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} - \frac{\widehat{y}_{k+1}(1)}{y_k(1)},$$

$$d_1 = -\frac{\omega^{V}(\lambda_k)}{20\omega'''(\lambda_k)} - \frac{\widehat{y}_{k+1}(1)\omega^{IV}(\lambda_k)}{4y_k(1)\omega'''(\lambda_k)} - \frac{\widehat{y}_{k+2}(1)\omega'''(\lambda_k)}{y_k(1)\omega'''(\lambda_k)} + c_2^2,$$

and $\widehat{y}_{k+2} = y_{k+2} - \widetilde{c}\widehat{y}_{k+1} - \widetilde{d}y_k$.

3. Affine case

Theorem 1 below was given in [**2**] and mentioned in [**12**]. Let $a = 0$. The case $b \geq 0$ is well known and was studied in many papers before (see e.g. [**11**], [**13**]). So, we assume in this section that $b < 0$.

Theorem 1. Let $a = 0$ and $b < 0$.

1) In the case **(c)** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k)$ forms a minimal system in $L_2(0,1)$ if and only if $y^*_{k+1}(1) \neq 0$, which is equivalent to

$$\widetilde{c} \neq \frac{1}{3} \cdot \frac{\omega'''(\lambda_k)}{\omega''(\lambda_k)}.$$

2) In the case **(d)** the system $\{y_n\}$ $(n=0,\ 1,\dots;\ n\neq k+1)$ forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+1}(1)\neq 0$, which is equivalent to

$$\widetilde{c}\neq \frac{1}{4}\cdot\frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}.$$

3) In the case **(d)** the system $\{y_n\}$ $(n=0,\ 1,\dots;\ n\neq k)$, forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+2}(1)\neq 0$, which is equivalent to

$$\widetilde{d}\neq \frac{1}{4}\cdot\frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\cdot\left(\widetilde{c}-\frac{1}{4}\cdot\frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right)+\frac{1}{20}\cdot\frac{\omega^{V}(\lambda_k)}{\omega'''(\lambda_k)}.$$

In each case the minimality conditions are also conditions to be a basis in $L_p(0,1)$ $(1<p<\infty)$ for the corresponding systems (see [**2**]).

Similar problem was also studied in [**1**]. In more general form these basis properties (minimality and completeness) of differential operators with eigenparameter dependent boundary conditions were studied in [**23**]. In particular, the results mentioned in Theorem 3 of [**22**] for the cases **(c)** and **(d)** above are in perfect agreement with Theorem 1 of the current paper.

4. Quadratic case

In the current section, more general results will be proved for the quadratic case and Theorem 1 above is a consequence of Theorem 2 (part 2) and Theorem 3 below.

Theorem 2. Let $a\neq 0$.

1) In the case **(c)** the system $\{y_n\}$ $(n=0,\ 1,\dots;\ n\neq k,j)$, where $j\neq k,k+1$ is arbitrary non-negative integer, forms a minimal system in $L_2(0,1)$ if and only if $y^{*}_{k+1}(1)(\lambda_j-\lambda_k)\neq y_k(1)$, which is equivalent to

$$\widetilde{c}\neq\frac{1}{\lambda_j-\lambda_k}+\frac{1}{3}\cdot\frac{\omega'''(\lambda_k)}{\omega''(\lambda_k)}.$$

In the case **(d)** the system

2) $\{y_n\}$ $(n=0,\ 1,\dots;\ n\neq k,k+2)$ forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+1}(1)\neq 0$, which is equivalent to

$$\widetilde{c}\neq \frac{1}{4}\cdot\frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)};$$

3) $\{y_n\}$ $(n=0,\ 1,\dots;\ n\neq k,k+1)$ forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+1}(1)^2\neq y_k(1)y^{\#}_{k+2}(1)$, which is equivalent to

$$\widetilde{d}\neq\widetilde{c}\cdot\left(\widetilde{c}-\frac{1}{4}\cdot\frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right)+\frac{1}{20}\cdot\frac{\omega^{V}(\lambda_k)}{\omega'''(\lambda_k)};$$

4) $\{y_n\}$ $(n=0,\ 1,\dots;\ n\neq k+1,j)$, where $j\neq k,k+1,k+2$ is arbitrary non-negative integer, forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+1}(1)(\lambda_j-\lambda_k)\neq y_k(1)$, which is equivalent to

$$\widetilde{c}\neq\frac{1}{\lambda_j-\lambda_k}+\frac{1}{4}\cdot\frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)};$$

5) $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, j)$, where $j \neq k, k+1, k+2$ is arbitrary non-negative integer, forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+2}(1)(\lambda_j - \lambda_k) \neq y^{\#}_{k+1}(1)$, which is equivalent to

$$\widetilde{d} \neq \left(\frac{1}{\lambda_j - \lambda_k} + \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) \cdot \left(\widetilde{c} - \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) + \frac{1}{20} \cdot \frac{\omega^{V}(\lambda_k)}{\omega'''(\lambda_k)}.$$

Proof. The part of the theorem about minimality in $L_2(0,1)$ was proved in [**4**] (See also [**16**]). So, we will prove only the equivalency of the two inequalities in each case.

1) Let us write the first inequality as $\frac{y^*_{k+1}(1)}{y_k(1)} \neq \frac{1}{\lambda_j - \lambda_k}$. By Definition 1 (Lemma 3.1 in [4]), $y^*_{k+1} = y_{k+1} + c_1 y_k$, where

$$c_1 = -\frac{\omega'''(\lambda_k)}{3\omega''(\lambda_k)} - \frac{y_{k+1}(1)}{y_k(1)} + \widetilde{c}.$$

So, we obtain $\frac{y_{k+1}(1)}{y_k(1)} + c_1 = -\frac{\omega'''(\lambda_k)}{3\omega''(\lambda_k)} + \widetilde{c} \neq \frac{1}{\lambda_j - \lambda_k}$, which completes the proof for this case.

2) By Definition 2 (Lemma 3.2 in [**4**]), $y^{\#}_{k+1} = y_{k+1} + c_2 y_k$, where

$$c_2 = -\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} - \frac{y_{k+1}(1)}{y_k(1)} + \widetilde{c}.$$

Then the inequality $y^{\#}_{k+1}(1) \neq 0$ is equivalent to

$$\frac{y_{k+1}(1)}{y_k(1)} + c_2 = -\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c} \neq 0,$$

which completes the proof for this case.

3) By Definition 2 (Lemma 3.3 in [**4**]),

$$d_1 - c_2^2 = -\frac{\omega^{V}(\lambda_k)}{20\omega'''(\lambda_k)} - \frac{1}{4}\left(\frac{y_{k+1}(1)}{y_k(1)} - \widetilde{c}\right)\frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)} - \frac{y_{k+2}(1)}{y_k(1)} + \widetilde{c}\frac{y_{k+1}(1)}{y_k(1)} + \widetilde{d} - \widetilde{c}^2.$$

Let us write the inequality as $\frac{y^{\#}_{k+1}(1)}{y_k(1)} \neq \frac{y^{\#}_{k+2}(1)}{y^{\#}_{k+1}(1)}$. So, we can write

$$\frac{y_{k+1}(1)}{y_k(1)} + c_2 \neq \frac{y_{k+2}(1) + (d_1 - c_2^2)y_k(1)}{y_{k+1}(1) + c_2 y_k(1)} + c_2,$$

or

$$\frac{y_{k+1}(1)}{y_k(1)} \neq \frac{\frac{y_{k+2}(1)}{y_k(1)} + (d_1 - c_2^2)}{\frac{y_{k+1}(1)}{y_k(1)} + c_2}.$$

Noting the expressions for c_2 and $d_1 - c_2^2$, we obtain

$$\frac{y_{k+1}(1)}{y_k(1)} \neq \frac{-\frac{\omega^{V}(\lambda_k)}{20\omega'''(\lambda_k)} - \frac{1}{4}\left(\frac{y_{k+1}(1)}{y_k(1)} - \widetilde{c}\right)\frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)} + \widetilde{c}\frac{y_{k+1}(1)}{y_k(1)} + \widetilde{d} - \widetilde{c}^2}{-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c}}.$$

Reorganizing the numerator of the right hand side, we obtain

$$\frac{y_{k+1}(1)}{y_k(1)} \neq \frac{-\frac{\omega^{V}(\lambda_k)}{20\omega'''(\lambda_k)} + \widetilde{d}}{-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c}} + \left(\frac{y_{k+1}(1)}{y_k(1)} - \widetilde{c}\right),$$

which completes the proof for this case.

4) Let us write the inequality as $\frac{y^{\#}_{k+1}(1)}{y_k(1)} \neq \frac{1}{\lambda_j-\lambda_k}$. Using the method of the previous cases, we can write

$$\frac{y_{k+1}(1)}{y_k(1)} + c_2 \neq \frac{1}{\lambda_j - \lambda_k},$$

which simplifies to

$$-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c} \neq \frac{1}{\lambda_j - \lambda_k}.$$

5) Let us write the inequality as $\frac{y^{\#}_{k+2}(1)}{y^{\#}_{k+1}(1)} \neq \frac{1}{\lambda_j-\lambda_k}$. We already simplified the left side of this inequality in the proof of the case **3**. Using it we obtain

$$\frac{y_{k+2}(1) + (d_1 - c_2^2)y_k(1)}{y_{k+1}(1) + c_2 y_k(1)} + c_2 \neq \frac{1}{\lambda_j - \lambda_k}.$$

Using the formula for c_2, we obtain

$$\frac{-\frac{\omega^V(\lambda_k)}{20\omega'''(\lambda_k)} + \widetilde{d}}{-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c}} + \left(\frac{y_{k+1}(1)}{y_k(1)} - \widetilde{c}\right) - \frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} - \frac{y_{k+1}(1)}{y_k(1)} + \widetilde{c} \neq \frac{1}{\lambda_j - \lambda_k},$$

or just

$$\frac{-\frac{\omega^V(\lambda_k)}{20\omega'''(\lambda_k)} + \widetilde{d}}{-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c}} - \frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} \neq \frac{1}{\lambda_j - \lambda_k},$$

which is equivalent to the required inequality.

Theorem 3. 1) In the case **(c)**, $y^*_{k+1}(1) \neq 0$ if and only if

$$\widetilde{c} \neq \frac{1}{3} \cdot \frac{\omega'''(\lambda_k)}{\omega''(\lambda_k)}.$$

2) In the case **(d)**, $y^{\#}_{k+2}(1) \neq 0$ if and only if

$$\widetilde{d} \neq \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)} \cdot \left(\widetilde{c} - \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) + \frac{1}{20} \cdot \frac{\omega^V(\lambda_k)}{\omega'''(\lambda_k)}.$$

Proof. **1)** Let us write the inequality as $\frac{y_{k+1}(1)}{y_k(1)} + c_1 \neq 0$. Using the formula for c_1, we obtain $\frac{y_{k+1}(1)}{y_k(1)} + c_1 = -\frac{\omega'''(\lambda_k)}{3\omega''(\lambda_k)} + \widetilde{c} \neq 0$, which completes the proof for this case.

2) Let us write the inequality as $\frac{y_{k+2}(1)}{y_k(1)} + c_2\left(\frac{y_{k+1}(1)}{y_k(1)} + c_2\right) + d_1 - c_2^2 \neq 0$. Using the formula for c_2 and d_1, we obtain

$$\frac{y_{k+2}(1)}{y_k(1)} + c_2\left(-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c}\right)$$
$$+\left(-\frac{\omega^V(\lambda_k)}{20\omega'''(\lambda_k)} + \left(\frac{y_{k+1}(1)}{y_k(1)} - \widetilde{c}\right)\left(-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c}\right) - \frac{y_{k+2}(1)}{y_k(1)} + \widetilde{d}\right) \neq 0$$

or

$$c_2\left(-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c}\right)$$
$$-\frac{\omega^V(\lambda_k)}{20\omega'''(\lambda_k)} + \left(-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} - c_2\right)\left(-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c}\right) + \widetilde{d} \neq 0.$$

Finally, after cancelling the terms with c_2, we obtain

$$-\frac{\omega^V(\lambda_k)}{20\omega'''(\lambda_k)} + \frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)}\left(-\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} + \widetilde{c}\right) + \widetilde{d} \neq 0,$$

which proves the theorem.

Remark: As in the previous section, one can extend these minimality conditions to basis properties in $L_p(0,1)$ $(1 < p < \infty)$ using the asymptotic formula for the eigenvalues and the fact that the corresponding systems of eigenfunctions are quadratically close to some trigonometric systems (see e.g. [**4**]).

5. Examples

In [**1**], [**2**], [**3**] many examples of the affine case were given. In those examples both ways of describing the necessary and sufficient conditions were presented. In all of these examples the two ways gave coinciding answers. So, the examples about the affine case will not be discussed here. Now we will use the new results in the current paper to do the same for the quadratic case. The examples below were previously studied only with one method. For completeness, we included those studies here and compared them with the new alternative method.

First example of a double eigenvalue. The following example from [**4**] was also discussed in [**14**], [**15**], [**16**]. Consider the spectral problem

$$-y'' = \lambda y,\ 0 < x < 1,$$
$$y'(0) = 0,\ y'(1) = \left(\frac{\lambda^2}{\pi^2} - \lambda\right) y(1).$$

For this problem

$$\omega(\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda} - \left(\frac{\lambda^2}{\pi^2} - \lambda\right)\cos\sqrt{\lambda}.$$

and

$$\omega(0) = \omega'(0) = 0,\quad \omega''(0) = -\frac{2}{3} - \frac{2}{\pi^2},\quad \omega'''(0) = \frac{1}{5} + \frac{3}{\pi^2},$$
$$\omega(\pi^2) = 0 \neq \omega'(\pi^2).$$

In [**4**] we applied part 1 of Theorem 4.1 to this example. It was shown that the system

$$\left\{-\frac{x^2}{2} + c,\quad \cos\sqrt{\lambda_i}x\quad (i = 3, 4, 5, \ldots)\right\},$$

is minimal in $L_2(0,1)$ if and only if

$$c \neq \frac{\pi^2 + 15}{10(\pi^2 + 3)} - \frac{1}{\pi^2}.$$

Let us now solve the same problem using the new method. By Theorem 2, the condition $y_1^*(1)(\lambda_2 - \lambda_0) \neq y_0(1)$ is equivalent to

$$\widetilde{c} \neq \frac{1}{\lambda_2 - \lambda_0} + \frac{\omega'''(0)}{3\omega''(0)} = \frac{1}{\pi^2 - 0} + \frac{1}{3}\cdot\frac{\frac{1}{5} + \frac{3}{\pi^2}}{-\frac{2}{3} - \frac{2}{\pi^2}} =$$
$$= \frac{1}{\pi^2} - \frac{\pi^2 + 15}{10(\pi^2 + 3)}.$$

Since $c=-\widetilde{c}$, we obtain again

$$c\neq\frac{\pi^2+15}{10(\pi^2+3)}-\frac{1}{\pi^2}.$$

Second example of a double eigenvalue. The following example is taken from [**3**]. Consider the spectral problem

$$-y''=\lambda y,\ 0<x<1,$$

$$y(0)=0,\ y'(1)=\left(\frac{\lambda^2}{4\pi^2}-\frac{5}{8}\lambda+\frac{9\pi^2}{64}\right)y(1).$$

For this problem

$$y(x,\lambda)=\sin\sqrt{\lambda}x,\quad y'(x,\lambda)=\sqrt{\lambda}\cos\sqrt{\lambda}x,$$

and

$$\omega(\lambda)=\sqrt{\lambda}\cos\sqrt{\lambda}-\left(\frac{\lambda^2}{4\pi^2}-\frac{5}{8}\lambda+\frac{9\pi^2}{64}\right)\sin\sqrt{\lambda}.$$

Here $\lambda=0$ is not an eigenvalue, $\lambda_0=\lambda_1=\frac{\pi^2}{4}$ is the double eigenvalue, $\lambda_2=\frac{9\pi^2}{4}$ and all other eigenvalues $\lambda_3<\lambda_4<\ldots$ are simple. Note that $y_0=\sin\frac{\pi}{2}x$, $y_2=\sin\frac{3\pi}{2}x$, $y_i=\sin\sqrt{\lambda_i}x$ $(i\geq 3)$, and

$$\widetilde{y}_1(x)=\lim_{\lambda\to\frac{\pi^2}{4}}y_\lambda(x,\lambda)=\lim_{\lambda\to\frac{\pi^2}{4}}\left(\frac{x\cos\sqrt{\lambda}x}{2\sqrt{\lambda}}\right)=\frac{x}{\pi}\cdot\cos\frac{\pi}{2}x.$$

So, we define the general first order associated function by $y_1=\frac{x}{\pi}\cdot\cos\frac{\pi}{2}x+c\cdot\sin\frac{\pi}{2}x$, that is $c=-\widetilde{c}$ (c is a constant). Let $y_1^*=\frac{x}{\pi}\cdot\cos\frac{\pi}{2}x+c'\cdot\sin\frac{\pi}{2}x$. From Lemma 5 it follows that $c'=\frac{4}{3\pi^2}-\frac{2}{9}-c$, so

$$y_1^*=\frac{x}{\pi}\cdot\cos\frac{\pi}{2}x+\left(\frac{4}{3\pi^2}-\frac{2}{9}-c\right)\cdot\sin\frac{\pi}{2}x.$$

The system

$$\left\{\frac{x}{\pi}\cdot\cos\frac{\pi}{2}x+c\cdot\sin\frac{\pi}{2}x,\quad \sin\sqrt{\lambda_i}x\quad(i=3,4,5,\ldots)\right\},$$

that is the system of root functions without the removed functions y_0 and y_2, is minimal in $L_2(0,1)$ if and only if $y_1^*(1)(\lambda_2-\lambda_0)\neq y_0(1)$, or more explicitly, if $c\neq\frac{5}{6\pi^2}-\frac{2}{9}$. If $c=\frac{5}{6\pi^2}-\frac{2}{9}$ then the function $g(x)=2\pi x\cdot\cos\frac{\pi}{2}x+\sin\frac{\pi}{2}x+\sin\frac{3\pi}{2}x$, is orthogonal to all the elements of the system. By Theorem 2, the condition $y_1^*(1)(\lambda_2-\lambda_0)\neq y_0(1)$ is equivalent to

$$\widetilde{c}\neq\frac{1}{\lambda_2-\lambda_0}+\frac{\omega'''(\frac{\pi^2}{4})}{3\omega''(\frac{\pi^2}{4})}=\frac{1}{\frac{9\pi^2}{4}-\frac{\pi^2}{4}}+\frac{1}{3}\cdot\frac{\frac{6-\pi^2}{\pi^4}}{-\frac{3}{2\pi^2}}=-\frac{5}{6\pi^2}+\frac{2}{9}.$$

Since $c=-\widetilde{c}$, we obtain again $c\neq\frac{5}{6\pi^2}-\frac{2}{9}$.

First example of a triple eigenvalue. The following example is taken from [**4**] and it was also discussed in [**20**]. Consider the problem

$$-y''=\lambda y,\ 0<x<1,$$

$$y'(0)=0,\ y'(1)=\left(-\frac{\lambda^2}{3}-\lambda\right)y(1).$$

For this problem

$$\omega(\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda} + \left(\frac{\lambda^2}{3} + \lambda\right)\cos\sqrt{\lambda},$$

$$\omega(0) = \omega'(0) = \omega''(0) = 0,\ \omega'''(0) = -\frac{4}{5},$$

$$\omega^{IV}(0) = \frac{32}{105},\ \omega^{V}(0) = -\frac{10}{189}.$$

We define the first and second order associated functions by $y_1 = -\frac{x^2}{2} + c$, $y_2 = \frac{x^4}{24} + c\left(-\frac{x^2}{2} + c\right) + d$, that is $c = -\widetilde{c}$, $d = -\widetilde{d}$ (c, d are constants). In [**4**] we have also shown that

$$y_1^{\#} = -\frac{x^2}{2} + \frac{25}{42} - c,\ y_2^{\#} = \frac{x^4}{24} - \left(\frac{25}{42} - c\right)\frac{x^2}{2} + \frac{1385}{5292} - \frac{25}{42}c - d.$$

In [**4**] we applied part 2 of Theorem 4.1 to this example and proved that the system

$$\left\{-\frac{x^2}{2} + c,\quad \cos\sqrt{\lambda_i}x \quad (i = 3, 4, 5, \ldots)\right\},$$

is minimal in $L_2(0,1)$ if and only if $y_1^{\#}(1) \neq 0$ or more explicitly if $c \neq \frac{2}{21}$. By Theorem 2, $y_1^{\#}(1) \neq 0$ if and only if

$$\widetilde{c} \neq \frac{\omega^{IV}(0)}{4\omega'''(0)} = \frac{1}{4}\cdot\frac{32/105}{-4/5} = -\frac{2}{21},$$

which gives the same result. Similarly, in [**4**] we applied part 3 of Theorem 4.1 to the system

$$\left\{\frac{x^4}{24} + c\left(-\frac{x^2}{2} + c\right) + d,\quad \cos\sqrt{\lambda_i}x \quad (i = 3, 4, 5, \ldots)\right\},$$

and proved that it is minimal in $L_2(0,1)$ if and only if $y_1^{\#}(1)^2 \neq y_0(1)y_2^{\#}(1)$ or more explicitly if $d \neq -\frac{5}{1512} + \frac{2}{21}c - c^2$.

By Theorem 2, $y_1^{\#}(1)^2 \neq y_0(1)y_2^{\#}(1)$ if and only if

$$\widetilde{d} \neq \widetilde{c}^2 - \frac{\widetilde{c}}{4}\cdot\frac{\omega^{IV}(0)}{\omega'''(0)} + \frac{1}{20}\cdot\frac{\omega^{V}(0)}{\omega'''(0)} =$$

$$= \widetilde{c}^2 - \frac{\widetilde{c}}{4}\cdot\frac{32/105}{-4/5} + \frac{1}{20}\cdot\frac{-10/189}{-4/5} = \widetilde{c}^2 + \frac{2}{21}\cdot\widetilde{c} + \frac{5}{1512},$$

which gives the same result.

Second example of a triple eigenvalue. The following example is taken from [**3**]. Consider the problem

$$-y'' = \lambda y,\ 0 < x < 1,$$

$$y(0) = 0,\ y'(1) = \left(-\frac{\lambda^2}{45} - \frac{\lambda}{3} + 1\right)y(1).$$

For this problem $y(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$, $y'(x,\lambda) = \cos\sqrt{\lambda}x$,

$$\omega(\lambda) = \cos\sqrt{\lambda} - \left(-\frac{\lambda^2}{45} - \frac{\lambda}{3} + 1\right)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}},$$

$$\omega(0) = \omega'(0) = \omega''(0) = 0,\ \omega'''(0) = -\frac{4}{315},$$

$$\omega^{IV}(0) = \frac{16}{4725},\ \omega^{V}(0) = -\frac{2}{4455}.$$

So, $\lambda_0 = \lambda_1 = \lambda_2 = 0$ is a triple eigenvalue, and $\lambda_3 < \lambda_4 < \dots$ are the positive solutions of the equation $\omega(\lambda) = 0$. Also, $y_0 = x,\ y_i = \sin\sqrt{\lambda_i}x\ \ (i \geq 3),\ y_1(x) = -\frac{x^3}{6} + cx$, and $y_2(x) = \frac{x^5}{120} - c\frac{x^3}{6} + dx$. We also find $c_2 = -2c + \frac{7}{30}, d_1 = \frac{329}{9900} - \frac{7}{15}c + 3c^2 - 2d$. Therefore, $y_1^{\#}(x) = -\frac{x^3}{6} + (-c + \frac{7}{30})x, y_2^{\#}(x) = \frac{x^5}{120} + (-c + \frac{7}{30})(-\frac{x^3}{6} - cx) + (\frac{329}{9900} - d)x$.

By part 2 of Theorem 4.1 in [**4**], the system

$$\left\{-\frac{x^3}{6} + cx,\ \ \sin\sqrt{\lambda_i}x\ \ (i = 3, 4, 5, \dots)\right\},$$

is minimal in $L_2(0,1)$ if and only if $y_1^{\#}(1) \neq 0$ or more explicitly if $c \neq \frac{1}{15}$. If $c = \frac{1}{15}$ then the function $\frac{x^5}{120} - \frac{x^3}{36} + \frac{7}{360}$ is orthogonal to all the elements of the system.

By Theorem 2, $y_1^{\#}(1) \neq 0$ if and only if

$$\widetilde{c} \neq \frac{\omega^{IV}(0)}{4\omega'''(0)} = \frac{1}{4} \cdot \frac{16/4725}{-4/315} = -\frac{1}{15}.$$

By part 3 of Theorem 4.1 in [**4**] the system

$$\left\{\frac{x^5}{120} + c\left(-\frac{x^3}{3} + c\right) + d,\ \ \sin\sqrt{\lambda_i}x\ \ (i = 3, 4, 5, \dots)\right\},$$

is minimal in $L_2(0,1)$ if and only if $y_1^{\#}(1)^2 \neq y_0(1)y_2^{\#}(1)$ or more explicitly if $d \neq -\frac{7}{3960} + \frac{1}{15}c - c^2$. If $d = -\frac{7}{3960} + \frac{1}{15}c - c^2$ then the same function $\frac{x^5}{120} - \frac{x^3}{36} + \frac{7}{360}$ is orthogonal to all the elements of the system.

By Theorem 2, $y_1^{\#}(1)^2 \neq y_0(1)y_2^{\#}(1)$ if and only if

$$\widetilde{d} \neq \widetilde{c}^2 - \frac{\widetilde{c}}{4} \cdot \frac{\omega^{IV}(0)}{\omega'''(0)} + \frac{1}{20} \cdot \frac{\omega^{V}(0)}{\omega'''(0)} =$$

$$= \widetilde{c}^2 - \frac{\widetilde{c}}{4} \cdot \frac{16/4725}{-4/315} + \frac{1}{20} \cdot \frac{-2/4455}{-4/315} = \widetilde{c}^2 + \frac{1}{15} \cdot \widetilde{c} + \frac{7}{3960},$$

which coincides with the previous result.

As these examples suggest, the alternative method, developed in the current paper, is simpler and gives the inequalities for minimality directly.

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